MILITARY TECHNICAL COLLEGE

CAIRO - EGYPT

1 # SIMPLIFICATION OF LINEAR DYNAMIC SYSTEMS

2

3 Mahmoud M. El-Metwally*

4 ## ABSTRACT

The modelling of large dynamic systems very often leads to linearized models of the form

$$\dot{X} = A X + B U$$

with the state-vector $X \varepsilon R^n$, the input-vector $U \varepsilon R^p$ and matrices A and B of appropriate dimensions. However, the largeness of a technical process often results in a high order n of the model . Therefore these models may be very difficult to employ for simulation or control design. To circumvent the drawback of high order, many authors suggest the application of order reduction techniques.

The previous work in the area of order reduction suffers from the following main drawbacks:

1. Most of the methods requires a priori knowledage of the behaviour of each state variable, which is not always known;
2. The level of parcentage error associated with order reduction is not indicated;
3. The effect of the imaginary part of sigenvalues has been neglected.

The present paper introduces a new method for simplification of linear dynamic systems. The level of percentage error is calculated and the effect of the imaginary part of the sigenvalues is considered. The determination of significant and less-significant state variables does not require a priori knowledege of the nature of the state variables. The method is tested through its application to a dynamic model of a synchonous machine .

---

5 * Associate Professor, Electrical Power & Machines Dept., Faculty of Engineering, Cairo University, Cairo Egypt.

INTRODUCTION

The modelling of large dynamic systems by the aid of physical laws very often leads to linearized models of the form

$$\dot{X} = A X + B U \qquad (1)$$

with the state-vector $X \in R^n$, the input-vector $U \in R^p$ and matrices A and B of appropriate dimensions. However, the largeness of a technical process often results in a high order n of the model (1). Therefore these models may be very difficult to employ for simulation or control design. To circumvent the drawback of high order, many anthors suggest the application of order reduction techniques.

The order reduction has been achieved by dividing the original system into a number of components and the reduction procedure is applied to each component separately [1,2,3]. The individual reduced order models are combined to get the final model. The order reduction of each component is achieved by the elimination of those state variables associated with the non-dominant eigenvalues (farthest from imaginary axis). The effect of imaginary part of the eigenvalues has been neglected.

Chidambara [4] obtained the reduced order system by dividing the original state-vector X in eqn 1 into dominant and non-dominant state vectors X_1 and X_2 , respectively. The reduced order system is obtained by setting $\dot{X}_2 = 0$. The method does not indicate how the order of X_1 is determined.

The Hurwitz polynomial approximation [5] and Routh approximation [6] have been used in order reduction of dynamic systems. This approach dos not give a definite rule for the determination of the order of the final reduced model.

Yu and El-Sharkawi [7] obtained the reduced order model using an iterative parameter estimation approach. An error cost function of a quadratic form has been employed. The final reduced order model depends on the weighting matrix contained in the cost function.

Verghese et al[8]developed a new approach termed "selective modal analysis". The method requires a priori Knowledge of system modes to be retained in the reduced order model, which is not always known.

The previous work in the area of order reduction suffers from the following main drawbacks:

1. Most of the methods require a priori knowledge of the behaviour of each state variable, which is not always known.
2. The level of percentage error associated with order reduction is not indicated.

6

3. The effect of the imaginary part of the eigenvalues has been neglected in most of the methods.

The present paper introduces a new method for order reduction of dynamic systems. The level of percentage error is calculated and the effect of the imaginary part of the eigenvalues is considered. The determination of significant and less-significant state variables does not require a priori knowledge of the nature of the state variables. The method is tested through its application to a six order model of a synchronous machine.

THE PROPOSED ORDER REDUCTION METHOD

Equation 1 can be rewritten in the following form:

$$\begin{bmatrix} \dot{X}_1 \\ \dot{X}_2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} U \qquad (2)$$

where X_1 is an m dimensional vector respresenting the significant state variables and X_2 and (n-m) dimensional vector repersenting the less significant state variables.

The Jordan transformation is given by:

$$X = P\ Y \qquad (3)$$

where P is the modal matrix of A containing n-independent eigenvectors corresponding to n distinct eigenvalues.

Equation 3 is rewritten in the form;

$$\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} \qquad (4)$$

From eqns. 2 and 4. we obtain ;

$$\dot{Y} = DY + RU$$

which is put in the form;

$$\begin{bmatrix} \dot{Y}_1 \\ \dot{Y}_2 \end{bmatrix} = \begin{bmatrix} D_1 & 0 \\ 0 & D_2 \end{bmatrix} \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} + \begin{bmatrix} R_1 \\ R_2 \end{bmatrix} U \qquad (5)$$

where,

$$\begin{bmatrix} D_1 & 0 \\ 0 & D_2 \end{bmatrix} = P^{-1} AP \text{ and } R = \begin{bmatrix} R_1 \\ R_2 \end{bmatrix} = P^{-1}B$$

The diagonal matrix D is given by;

$$D = \text{diag.}\ (\lambda_1\ ,\ \lambda_2\ ,\ldots,\ \lambda_n) \tag{6}$$

such that Re $(\lambda_i) \geq$ Re (λ_{i+1}), for $i=1,2,\ldots,(n-1)$

From eqn. 5, we have;

$$\dot{Y}_1 = D_1\ Y_1 + R_1\ U \tag{7}$$

$$\dot{Y}_2 = D_2\ Y_2 + R_2\ U$$

Assuming $\dot{Y}_2 = 0$, yields;

$$\bar{Y}_2 = -\ D_2^{-1}\ R_2\ U$$

where $\bar{Y}_2$ is the approximate value of Y_2. Neglecting the dynamics of Y_2 (i.e., $\dot{Y}_2 = 0$) produces an error in Y, which in turn leads to an error E in X as given by;

$$E = \begin{bmatrix} E_1 \\ E_2 \end{bmatrix} = \begin{bmatrix} X_1 - \bar{X}_1 \\ X_2 - \bar{X}_2 \end{bmatrix} = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \begin{bmatrix} 0 \\ Y_2 - \bar{Y}_2 \end{bmatrix} \tag{8}$$

where,

$$\begin{bmatrix} \bar{X}_1 \\ \bar{X}_2 \end{bmatrix} = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \begin{bmatrix} \bar{Y}_1 \\ \bar{Y}_2 \end{bmatrix} \tag{9}$$

$\bar{X}_1$, $\bar{X}_2$, $\bar{Y}_1$ and $\bar{Y}_2$ are the approximate values of X_1, X_2, Y_1 and Y_2, respectively.

Derivation of Absolute Error Formula

The norm of the error vector E (given by eqn. 8) is

$$\|E\| = \|P(Y_2 - \bar{Y}_2)\| \tag{10}$$

Using the norm properties , eqn. (10) becomes;

$$\|E\| \leq \|P\|\ \|Y_2 - \bar{Y}_2\| \tag{11}$$

where, $Y_2 = e^{D_2 t}\ Y_2(o) + \int_o^t e^{D_2(t-s)}\ R_2\ U(s)\ ds$

$$\bar{Y}_2 = -D_2^{-1} R_2 U \quad \text{and} \tag{12}$$

$$Y_2-\bar{Y}_2 = e^{D_2 t} Y_2(o) + \int_o^t e^{D_2(t-s)} R_2 U(s)\,ds + D_2^{-1} R_2 U \tag{13}$$

Assuming a constant input vector $U = U_o$ with $\| U_o \| = \mu$, gives;

$$\| Y_2-\bar{Y}_2 \| = \| e^{D_2 t} Y_2(o) + D_2^{-1} e^{D_2 t} R_2 U_o \|$$

$$\leq \| e^{D_2 t} \| \, \|Y_2(o)\| + \| D_2^{-1} \| \, \| e^{D_2 t} \| \, \| R_2 \| \, \| U_o \| \tag{14}$$

The norm of $e^{D_2 t}$ is given by

$$\| e^{D_2 t} \| = | e^{\sigma_{m+1} t} |$$

where σ_{m+1} is the real part of λ_{m+1}

The norm of D_2^{-1} is given by

$$\| D_2^{-1} \| = \frac{1}{\min|\lambda_i|} \quad , \; m+1 \leq i \leq n$$

Substituting for $\| e^{D_2 t} \|$ and $\| D_2^{-1} \|$ in eqn. 14, gives

$$\| Y_2-\bar{Y}_2 \| \leq | e^{\sigma_{m+1} t} | \left[\| Y_2(o) \| + \frac{\| R_2 \| \mu}{\min|\lambda_i| , \; m+1 \leq i \leq n} \right] \tag{15}$$

Substituting for $\| Y_2-\bar{Y}_2 \|$ from eqn. 15 into eqn.11, yields;

$$\| E \| \leq \| P \| \; | e^{\sigma_{m+1} t} | \left[\| Y_2(o) \| + \frac{\| R_2 \| \; \mu}{\min|\lambda_i| \; , \; m+1 \leq i \leq n} \right] \tag{16}$$

Normalization of the Error

The error norm is normalized to get the relative error with respect to the norm of the exact value of the state vector.

The Y and X vectors are related by:

$$Y = P^{-1} X$$

Taking the norm of both sides, gives;

$$\| Y \| \leq \| P^{-1} \| \; \| X \| \tag{17}$$

From eqns. 11 and 17, we obtain

$$\| E \| \; \| Y \| \leq \| P \| \; \| P^{-1} \| \; \| X \| \; \| Y_2-\bar{Y}_2 \| \tag{18}$$

Dividing both sides of eqn. 18 by $||X||\ ||Y||$, gives

$$\frac{||E||}{||X||} \leq ||P||\ ||P^{-1}||\ \frac{||Y_2-\bar{Y}_2||}{||Y||} \qquad (19)$$

The solution of eqn. 5 (for $U = U_o$), is given by;

$$Y(t) = e^{Dt}\ Y(o) + D^{-1}(e^{Dt} - I)\ RU_o$$

Taking the norm of both sides, gives

$$||Y|| = ||e^{Dt}\ Y(o) + D^{-1}(e^{Dt} - I)\, RU_o|| \qquad (20)$$

Substituting from eqns. 15 and 20 into eqn. 19, yields;

$$\frac{||E||}{||X||} \leq \text{cond.}(P)\ \frac{|e^{\sigma_{m+1}t}|\,[\,||Y_2(o)|| + \frac{||R_2||\ \mu}{\min|\lambda_i|,\ m+1\leq i\leq n}]}{||e^{Dt}\ Y(o) + D^{-1}(e^{Dt} - I)\, RU_o||} \qquad (21)$$

where cond $(P) = ||P||\ \ ||P^{-1}||$ and is called the condition number number of P.For certain values of m, $Y_2(o)$ and μ, the factor

$$[\,||Y_2(o)|| + \frac{||R_2||\ \ \mu}{\min|\lambda_i|,\ m+1\leq i\leq n}]$$ is constant with respect to time. On the other hand, the factor $e^{\sigma_{m+1}t}$ is continuously decreasing as t increases. The effect of the imaginary part of the eigenvalues is taken into account in the term $\min|\lambda_i|$, $m+1 \leq i \leq n$.

ORDER REDUCTION PREOCDURE

To dermine the order m of the reduced model, the bounded value of the relative error $||E||\,/\,||X||$ is calculated for different values of m (1,2,....,n) and for the time interval of interest. Fig.(1) gives a flow chart showing the main steps of the proposed order reduction procedure. The final state space form of the reduced model is given by;

$$\dot{X}_r = A_r X_r + B_r\ U$$

where X_r is m-dimensional vector containing the significant state variables. A_r and B_r are mxm and mxp coefficient matrices, respectively. The A_r and B_r matrices are given by;

$$A_r = P_{11}\ D_m\ P_{11}^{-1}, \qquad \text{and}$$

$$B_r = P_{11}^{-1}\ \ S$$

where

P_{11} is an mxm matrix defined by eqn. 4

D_m is a diagonal matrix containing the first m eigen-

Determine the eigenvalues and the corresponding eigenvectors of the matrix A

↓

Form the matrix $D=P^{-1}AP$ with $D=\text{diag.}(\lambda_1, \lambda_2, \ldots, \lambda_n)$ and $\text{Re}(\lambda_{i+1}) \leq \text{Re}(\lambda_i)$, $i = 1,2,\ldots,n-1$

↓

Calculate R_2 and $\left[\frac{1}{\min\limits_{m+1 \leq i \leq n} |\lambda_i|}\right]$ for all m from m = 1 up to m = n

↓

Evaluate % error $\frac{\|e\|}{\|x\|}$ for the different values of m and for the time interval of interest.

↓

Determine the value of m which satisfies the desired error level

↓

Calculate A_r, B_r where $A_r = P_{11} D_m P_{11}^{-1}$ and $B_r = P_{11}^{-1} S$

Fig. 1 Computation procedure of the proposed order reduction method.

values stored in the D matrix

S is an mxp matrix containing the first m rows of $P^{-1}B$ matrix.

ILLUSTRATIVE EXAMPLE

In order to illustrate the order reduction procedure the proposed method is applied to a six order dynamic system, of the form

$$\dot{X} = AX + RU$$

where the A and B matrices are given by

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & -23.4 & 37.4 & -53.48 & 88.1 \\ 235.9 & 0 & -43.2 & 39.25 & 431.75 \\ 0 & 0 & 0.98 & -1.18 & 0 \\ -207.2 & 0 & -637.8 & 294.4 & -15.7 \\ 4.414 & 0 & 18.69 & -16.98 & -3.21 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 314 \\ 0 \\ -2 \end{bmatrix} ,$$

$$B^t = [0\ 0\ 0\ 0\ 0\ 2\ ]$$

The eigenvalues of the A matrix are

$$\lambda_{1,2} = 1.72 \pm j\ 50.47$$
$$\lambda_{3} = -3.136$$
$$\lambda_{4} = -20.34$$
$$\lambda_{5,6} = -29.244 \pm j\ 527.127$$

Fig.2 shows the variation of the bounded value of the relative error $\|E\| / \|X\|$ for different values of m ($y(o) = 0$ and $\|U\| = 1$). It is seen that for a relative error of 5% and for m=3, the correponding time interval is $0.45 \leq t < \infty$. For the same error level and for m=4, the time interval becomes $0.05 \leq t \leq \infty$

Fig.3 shows the time response of X_1 for m=3, m=4 and m=n=6. It is seen that m=4 gives a time response close to that of the original system (m=n=6). Other state variables have shown similar dynamic behaviour.

CONCLUSION

A new method for order reduction of linear dynamic systems has been proposed. The method takes into account the effect of the imaginary part of the eigenvalues and the percentage value of the relative error associated with the reduced system.

The capability of the method has been tested through its application to a six order linear dynamic system.

6

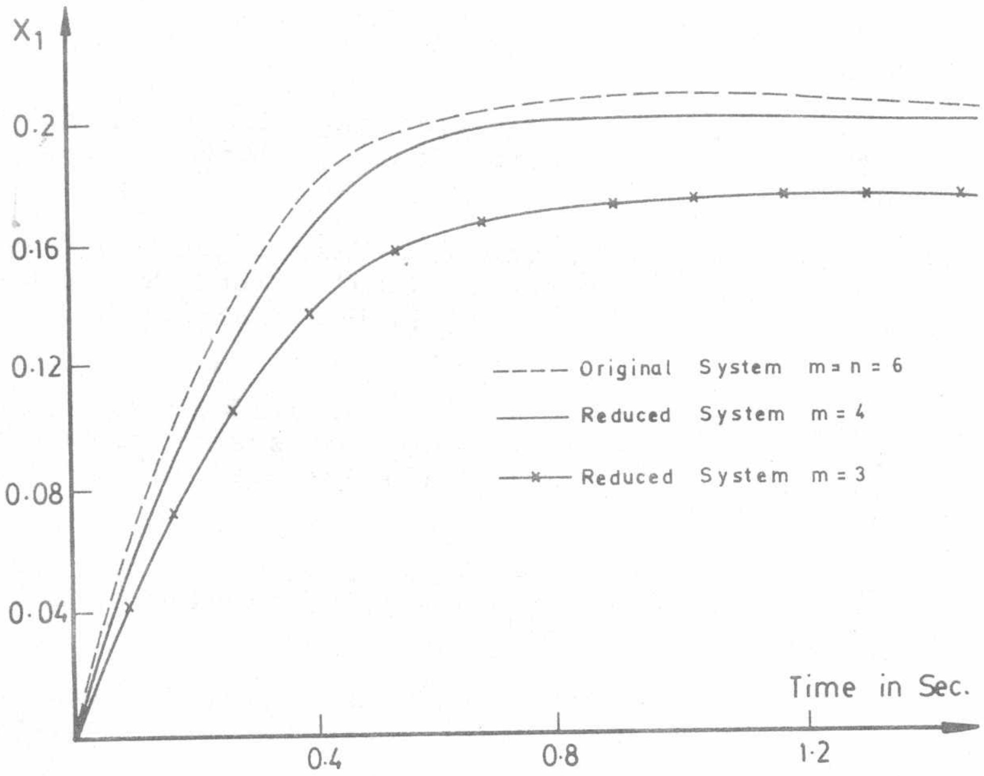

Fig. (2) Time Response for U(t) = Unit Step

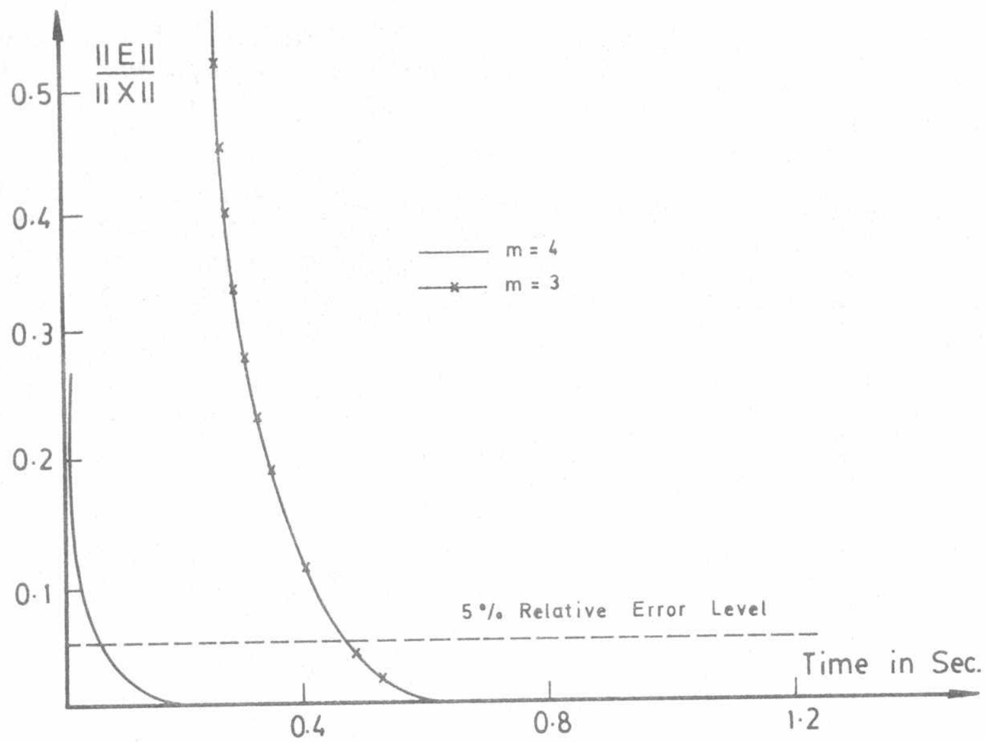

Fig. (3) Relative Error

REFERENCES

1. DAVISON,E.J. :'A method for simplifing linear dynamic systems' IEEE Trans. on Automatic Control, 1966, Vol. AC-11, pp. 93-101.

2. KAPPURAJULU,A., and ELANGOVAN,S.: 'Simplified Power system models for dynamic stability studies'., IEEE Trans. on Power Apparatus and systems, 1971, Vol. PAS-90, No. 1, pp. 11-23.

3. ALTALIB, H.Y., and KRAUSE, P.C. : 'Dynamic equivalents by combination of reduced order models of system components'. IEEE Trans. on Power Apparatus and Systems, 1976, Vol. PAS-95, No. 5, pp. 1535-1544.

4. CHIDAMBARA, M.R. : "Two simple techniques for the simplification of large dynamic systems", Proceeding of J.A.C.C., 1969, pp. 669-674.

5. APPIAH, R.K., 'Linear model reduction using Harwitz polynomial approximation', Int. J. of Control, 1978, Vol.28, No.3, pp. 476-488.

6. HUTTON, M.F.,and FRIEDLAND, B. 'Routh approximation for reducing order of linear of linear time-invariant systems'., IEEE Trans.on Automatic Control, 1975, Vol. AC. 20,pp. 320-337.

7. Yu,Y., and EL-SHARKAWI, M.A.: 'Estimation of external dynamic equivalents of a thirteen machine system'., IEEE Trans, on Power Appratus and systems, 1981, Vol. PAS-100, No. 3, pp. 1324-1332.

8. VERGHESE, G.C., PEREZ-ARRIAGA, I.J., and SCHWEPPE, F.C.: 'Selective modal analysis with application to electric Power systems. Part II : The dynamic stability problem' IEEE Winter Power Meeting, N.Y., 1982, Paper No. 82-WM 071-9.