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A DESIGN PROCEDURE FOR PI OUTPUT-FEEDBACK MODAL CONTROLLERS

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ABSTRACT

In this paper the problem of pole-assignment in linear time invariant multivariable systems is generalized to include the case of systems incorporating integral control action. In some control system design poroblems, integral feedback is introduced in conjunction with proportional one in order to eleminate steady-state errors arising from sustained disturbances.
Most published work deals with the introduction of integral as well as proportional feedback of some or all of the state variables of the system under control. This paper establishes a new design procedure of proportional plus-integral controllers (PI-controllers) based on the feedback of system output variables.
The controllability of the compensated system is investigated and a necessary and sufficient condition for the augmented system to be controllable is derived. The existance problem of such type of modal-controllers is discussed and a necessary condition for complete pole-assignment of the closed-loop system is derived.
A pratical design algroithm of this type of controllers is developed, and it is used to design a PI-controller for a VSTOL aircraft.

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1. INTRODUCTION

The problem of pole-assignment using integral feedback control in conjunction with proportional one was investigated by Porter and Power [5] . Porter and Crossley [6] have shown that if the feedback control matrix for proportional-plus-integral control is a single dyadic product, then the integral of no more than one state variable can be fed back. Park and Seborg [7] tried to bypass this limitation by feeding q-state variables, providing that the integral control matrix has full rank q, and $q \leq m$, where m is the dimension of the control vector. These design procedures ([5]- [7]) were based on the application of the system state-feedback.
In the present paper we introduce a new design procedure of PI-modal controllers using the feedback of system output vector, where deep insight of the problem is given, and the existance problem of such type of modal-controllers is discussed in more detail. In sec.2, theoritical formulation of the problem is estabilished and the main results of this paper are given in the form of a lemma and a proposition. A practical design algorithm is developed in sec.3, and it is applied to design a PI-controller for a VSTOL aircraft in sec.4.

2. THEORITICAL ANALYSIY

2.1. Compensation Problem

Consider the controllable and observable linear-multivariable system governed by the following equations

$$\dot{x}(t) = A\,x(t) + B\,u(t) \qquad (1.a)$$

$$y(t) = C\,x(t) \qquad (1.b)$$

where x(t) is the nxl state-vector, u(t) is the mxl control-vector, y(t) is the ℓxl output - vector, A,B, and C are constant real matrices of appropriate dimensions, matrices B and C are of full rank m,ℓ respectively. A is assumed initially to be non-singular, and $\ell \leq m$.
Suppose that it is required to eliminate the steady-state errors among the system output variables.Hence, the integral of the output vector is to be fedback, and the resulting state vector will consist of x(t) augmented by the ℓxl vector $\xi(t)$, where

$$\dot{\xi}(t) = -\,y(t) + y_c \qquad (2)$$

y_c is the ℓxl constant steady-state value of the output vector . Eqn.(2) can alternatively be written as

$$\dot{\xi}(t) = -C\,x(t) + y_c \qquad (3)$$

The state and output equations of the augmented system are then clearly

$$\begin{bmatrix} \dot{x}(t) \\ \dot{\xi}(t) \end{bmatrix} = \begin{bmatrix} A & 0 \\ -C & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ \xi(t) \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u(t) + \begin{bmatrix} 0 \\ I_\ell \end{bmatrix} y_c \qquad (4.a)$$

$$\begin{bmatrix} y(t) \\ \xi(t) \end{bmatrix} = \begin{bmatrix} C & 0 \\ 0 & I_\ell \end{bmatrix} \begin{bmatrix} x(t) \\ \xi(t) \end{bmatrix} \qquad (4.b)$$

where I_ℓ is the identity matrix of dimension $\ell x \ell$.
Introducing the following abbreviations:

$$\bar{x}(t) = \begin{bmatrix} x(t) \\ \xi(t) \end{bmatrix} ;(n+\ell) \times 1 \quad \text{state-vector} \tag{5.a}$$

$$\bar{y}(t) = \begin{bmatrix} y(t) \\ \xi(t) \end{bmatrix} ;2\ell \times 1 \quad \text{output-vector} \tag{5.b}$$

and

$$\bar{A} = \begin{bmatrix} A & 0 \\ -C & 0 \end{bmatrix} ;(n+\ell)\times(n+\ell) \text{ plant-matrix} \tag{6.a}$$

$$\bar{B} = \begin{bmatrix} B \\ 0 \end{bmatrix} ;(n+\ell) \times m \quad \text{control-matrix} \tag{6.b}$$

$$\bar{C} = \begin{bmatrix} C & 0 \\ 0 & I_\ell \end{bmatrix} ;2\ell \times (n+\ell) \text{ output-matrix} \tag{6.c}$$

$$R = \begin{bmatrix} 0 \\ I_\ell \end{bmatrix} ; (n+\ell) \times \ell \quad \text{matrix} \tag{6.d}$$

Then, eqn.(4) can be written as

$$\dot{\bar{x}}(t) = \bar{A}\ \bar{x}(t) + \bar{B}\ u(t) + R\ y_c \tag{7.a}$$

$$\bar{y}(t) = \bar{C}\ \bar{x}(t) \tag{7.b}$$

2.2. Controllability and Compensation

It was shown that [6] the introduction of the integral control action may destroy the controllability of the system. From the mode-controllability matrix of the augmented system-eqn.(7)-, we can reach the following result

Lemma

The system (7) will be controllable if and only if the system (1) is controllable, and

$$\text{rank}\left[CA^{-1}B\right] = \ell \tag{8.a}$$

Proof

The proof is straight forward and it can be found in a similar way to that one introduced in [6]. It will be omitted here for brevity.

It is necessary to notice that the preassumption of matrix A to be non-singular is not a limitation, since we can initially introduce a constant output feedback such that the plant-matrix (A-BKC) will possess n non-zero eigenvalues, where K is an $m\times\ell$ constant output feedback matrix.

Then , the above mentioned lemma can be rewritten as: The system (7) will be controllable if and only if the following conditions are satisfied:

1. the system (1) is controllable;
2. there exists a matrix K such that the matrix (A-BKC) is non-singular;
3. the ℓ -rows of the matrix $\left[C(A-BKC)^{-1}B\right]$ are linearly independent.In other words

$$\text{rank}\left[C(A-BKC)^{-1}B\right] = \ell \tag{8.b}$$

Remark (1)

If the matrix A is a singular one, we can easily find a matrix K such that $\hat{A} = A-BKC$ is non-singular. In the case that A is non-singular matrix, take $K = 0$, and $\hat{A} = A$. In all the following equations and expressions, the matrix $\hat{A}$ will replace the matrix A.

Remark (2)

Condition (3) of the lemma - expressed by eqn.(8.b)-can equivalently be written as [7]

$$\text{rank}\begin{bmatrix} A & B \\ -C & 0 \end{bmatrix} = n+\ell \qquad (8.c)$$

2.3. Existance Conditions

Now, the problem is to design a proportional-plus-integral output-feedback controller, having the control law

$$u(t) = -K_1\, y(t) - K_2\, \xi(t) \qquad (9)$$

such that the resultant closed-loop system has a specified set of stable poles, where K_1 is the proportional output-feedback matrix (of dimension $m \times \ell$), and K_2 is the integral output-feedback matrix (of dimensions $m\times\ell$). The following proposition provides a statement of the additional restriction which the matrix K_2 must satisfy when a control law of the form (9) is applied.

Proposition

Having the system (7) to be completely controllable, then a linear feedback control law of the form (9) shifts all the $(n+\ell)$ eigenvalues of the plant matrix $\bar{A}$ such that they will be arbitrarily close to $(n+\ell)$ preassigned non-zero value (subject to complex pairing) If the rank of the $m \times \ell$ matrix K_2 is equal to ℓ , where $\ell \leq m$ as it was assumed before.

Proof

We shall prove that if rank $K_2 < \ell$, then the modes (eigenvalues) of the closed - loop system matrix are not completely controllable.

Apply the control-law (9), to the system (7), the composite state-equation will be

$$\dot{\bar{x}}(t) = \bar{A}\,\bar{x}(t) - \bar{B}\begin{bmatrix} K_1 & K_2 \end{bmatrix}\bar{y}(t) + R\, y_c$$

or

$$\dot{\bar{x}}(t) = \bar{A}\,\bar{x}(t) - \bar{B}\begin{bmatrix} K_1 & K_2 \end{bmatrix}\bar{C}\,\bar{x}(t) + R\, y_c \qquad (10)$$

Now, the closed-loop system matrix will be given by

$$A_c = \bar{A} - \bar{B}\begin{bmatrix} K_1 & K_2 \end{bmatrix}\bar{C}$$

or

$$A_c = \begin{bmatrix} A & 0 \\ -C & 0 \end{bmatrix} - \begin{bmatrix} B \\ 0 \end{bmatrix}\begin{bmatrix} K_1 & K_2 \end{bmatrix}\begin{bmatrix} C & 0 \\ 0 & I_\ell \end{bmatrix} \qquad (11)$$

which can be rewritten as

$$A_c = \left[\begin{array}{c|c} A-BK_1C & -BK_2 \\ \hline -C & 0 \end{array}\right] = \left[\begin{array}{c|c} A_{c1} & A_{c2} \end{array}\right] \qquad (12)$$

Let rank $K_2 = \hat{\ell}$; $\hat{\ell} < \ell$. In this case, the right block matrix A_{c2} in eqn. (12) will have $(\ell - \hat{\ell})$ linearly dependent columns, which means that the matrix A_c has at least $(\ell - \hat{\ell})$ zero-eigenvalues, i.e. there are at least $(\ell - \hat{\ell})$ poles of the closed - loop system which cannot be altered by applying feedback. This means that the system is not completely controllable which contradicts the initial assumption. Therefore, the matrix K_2 must be of full rank ℓ .
This proposition gives a necessary, but not a sufficient condition for arbitrary pole-placement using PI-controllers.

2.4. Form of Controller

Now, we can rewrite eqn.(9) as

$$u(t) = -\begin{bmatrix} K_1 & K_2 \end{bmatrix} \begin{bmatrix} y(t) \\ \xi(t) \end{bmatrix} \qquad (13)$$

or

$$u(t) = -\bar{K}\,\bar{y}(t) \qquad (14)$$

where

$$\bar{K} = \begin{bmatrix} K_1 & K_2 \end{bmatrix} \qquad (15)$$

We see from eqn.(14) that the problem of designing a PI-modal controller is reduced to find an equivalent proportional output-feedback controller $\bar{K}$- where $\bar{K}$ is an $(m \times 2\ell)$ constant matrix - in order that the $(n+\ell)$ poles of the closed-loop system are arbitrarily close (but not necessarily equal) to $(n+\ell)$ preassigned values.
As it is known that the complete pole-assignment problem via constant output-feedback still remains unresolved, we have the following alternative approaches for constructing the required matrix $\bar{K}$. Different cases may arise

(1) For $(m + 2\ell - 1) \geq (n+\ell)$, complete pole-placement can be achieved, using for example the algorithm of [2] or [12] .
(2) For $(m+2\ell-1) < (n+\ell)$, we can try to use the algorithm of [3] or [8] or [9] .
(3) Otherwise, we can work over the whole set of $(n+\ell)$ poles to be approximately assigned close to a desired set of prescribed locations in the complex plane using for example the algortihm of [1] directly, or the algorithm of [10] after a little adaptation to the present problem.

3. DESIGN ALGORITHM

The theoritical analysis of the problem introduced in the previous section leads to the following design algorithm.

1- The plant matrix A is checked to be non-singular. If A is singular, find a $m \times \ell$ constant matrix K of output-feedback such that the matrix $\hat{A}=A-BKC$ is non-singular. If A is non-singular, set K=0 and $\hat{A}=A$. Notice that $\hat{A}$ will replace A in all the following steps.
2- Check the controllability of the augmented system, eqn.(7), by verifying the condition (8.b) or the condition (8.c).
3- Construct the matrices $\bar{A}$, $\bar{B}$ and $\bar{C}$ according to eqn.(6).
4- Find the matrix $\bar{K}$, eqn.(14) , as explained in paragraph 2.4.

5- The obtained matrix $\bar{K}$ can be partitioned according to eqn.(15) to get the matrices K_1 and K_2.
6- Resultant proportional output-feedback matrix K_r will be

$$K_r = K_1 + K \tag{16}$$

4. PRACTICAL APPLICATION

We shall consider the problem of pole-assignment of a VSTOL aircraft using a PI-modal controller. The dynamics of the uncontrolled aircraft can be modelled by an equation of the form (1) , where the state vector, x(t) , the input vector, u(t), the matrices A,B, and C are given as (c.f. [6])

$$x(t)=\begin{bmatrix}\beta(t)\\ p(t)\\ r(t)\\ \phi(t)\end{bmatrix} \quad ; \quad u(t)=\begin{bmatrix}\psi(t)\\ \eta(t)\end{bmatrix} \tag{17}$$

$$A = \begin{bmatrix} -0.0506 & 0 & -1 & 0.238 \\ -0.7374 & -1.3345 & 0.3696 & 0 \\ -0.01 & 0.1074 & -0.3320 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \tag{18.a}$$

$$B = \begin{bmatrix} 0.0409 & 0 \\ 1.2714 & -20.3106 \\ -1.0625 & 1.335 \\ 0 & 0 \end{bmatrix} \tag{18.b}$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{18.c}$$

where

$\beta(t)$ = sideslip angle;
$p(t)$ = roll rate ;
$r(t)$ = yaw rate;
$\phi(t)$ = bank angle;
$\psi(t)$ = rudder angle ;
$\eta(t)$ = aileron angle.

The eigenvalues of A are the set [6]

$$\{0.047 \pm 0.3147\, i \;;\; -0.4023 \;;\; -1.4088\}$$

It is desired to find a PI-controller such that the six closed loop poles of the matrix A_c,eqn.(11) , will be located at

$$\{-0.1063 \pm 0.3717\, i \;;\; -0.1416 \pm 2.2029\, i \;;\; -0.6587 \pm 4.9412\, i\}$$

We shall proceed by following the steps of the algorithm described in sec.3.
1- It is clear that the matrix A is non-singular, hence K=0 , and $\hat{A}$=A.
2- The augmented systm (7), is completely controllable as

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$$\text{rank}\begin{bmatrix} A & B \\ -C & 0 \end{bmatrix} = 6 = n + \ell$$

3- We easily construct the matrices $\bar{A}$, $\bar{B}$ and $\bar{C}$ as given in eqn.(6) .

4- To find the equivalent matrix $\bar{K}$; eqn.(14) ; we follow-for example-the algorithm of [3] , and it is found to be

$$\bar{K} = \begin{bmatrix} 2.35 & -1.47 & 0.0126 & 2.7302 \\ 0.048 & -1.33 & 0.163 & 0.189 \end{bmatrix} \tag{19}$$

5- From eqn. (15) we get

$$K_1 = \begin{bmatrix} 2.35 & -1.47 \\ 0.048 & -1.33 \end{bmatrix} \tag{20a}$$

and

$$K_2 = \begin{bmatrix} 0.0126 & 2.7302 \\ 0.163 & 0.189 \end{bmatrix} \tag{20.b}$$

We find that the poles of the closed-loop system are exactly assigned.

5. CONCLUDING REMARKS

In this paper we have presented the control technique that makes use of integral as well as proportional feedback of the output variables of the system under control in order to assign the closed-loop system poles. We have tried to give deep insight of this modal control problem which has not yet been explored enough. Two main results appear in this paper :

1. a lemma which establishes the necessary and sufficient conditions for a system to remain controllable in the presence of feedback from the integral of its output variables;
2. a propositon giving the necessary condition for complete pole-assignment of the augmented system to be achieved using PI-controllers.

It is worthy to mention that after the development of the material of this paper, the author's attention was drawn to two papers which tackle with the same problem. The first one is due to Seraji [11], which presents a multi-stage frequency domain technique to design a PI-controller that assigns $(2m + \ell - 1)$ poles of the $(n+\ell)$ poles of the augmented system. The second one is due to Novin-Hirbod [4] , where a similar approach to that of [11] has been introduced to find a PI-controller that assigns $(2\ell+m-1)$ poles of the composite system. The desired controller is constructed from a sequence of dyads and it is carried out in the frequency domain. Here, an equivalent proposition to that of Sec.2. appears in the form of a lemma, but with a different proof, which-in fact-is similar to the proof given in [7] dealing with the case of PI-state feedback.

REFERENCES

1. Bengtsson, G. and Lindahl, S.," A design scheme for incomplete state or output-feedback", Automatica, 10,15 (1974).
2. Davison, E.J. and Wang, S.H.," A note on pole assignment in linear systems with incomplete state-feedback", IEEE Trans. Auto.Control, AC-20, 516 (1975)
3. Mady, M.S.,"Sur la conception de compensateurs modaux avec applications pratiques", Ph.D. Thesis.Paris IX-University, France (1980).

4. Novin-Hirbod, S., "Pole assignment using proportional-plus-integral output-feedback control" Int.J. Control, 29,1035 (1979).
5. Porter, B. and Power, H.M., "Dyadic modal control of multi-input time-invariant linear systems incorporating integral feedback", Electron. Lett., 7,18 (1971).
6. Porter, B. and Crossley, R., "Modal Control. Theory and applications" , Taylor & Francis Ltd, London (1972).
7. Park, H. and Seborg, D.E.," Eigenvalue assignment using proportional-integral feedback control", Int.J.Control, 20,517 (1974).
8. Paraskevopoulos, P.N. and Tzafestas, S.G., "New results in feedback modal-controller design", ibid., 21,911 (1975).
9. Paraskevopoulos, P.N.," A general solution to the output-feedback eigenvalue assignment problem", ibid ., 24,509 (1976).
10. Sirisena, H.R. and Choi, S.S., "Optimal pole placement in linear multivariable systems using dynamic output-feedback", ibid., 21,661 (1975).
11. Seraji, H.,"Design of proportional-plus-integral controllers for multivariable systems", ibid., 29 , 49 (1979).
12. Topalogler, T. and Seborg, D.E.," A design procedure for pole assignment using output-feedback", Int.J. System Sci., 8,251 (1977).