A ROTATING MASS-SPRING-DAMPER SYSTEM UNDER COMBINED HARMONIC AND RANDOM EXCITATION

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ABSTRACT

A vibratory system of mass-spring-damper rotates about an axis perpendicular to its plane of vibration, and is subjected to harmonic forces in that plane. By taking into consideration the various coupling terms that may be present, we can measure the small rate of turn around the axis perpendicular to the plane vibration from the amplitude and phase angle relations. It was found previously that the phase angle is independent of the damping factor (i.e. independent of the transient response) at the value of the rate of turn which is smaller than the system natural frequencies. Therefore, we can improve the transient performance without affecting the sensitivity of the devise.

In this work, we study the sensitivity of the system when it is subjected to random noise.

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Introduction

The purpose of this paper is to study the effect of random noise on the measurement of small rate of turn by a vibratory system.

This system, described by Linnett (1969), is used to measure a small rate of turn. It is a two degrees of freedom vibratory rate sensor which consists of a single mass mounted at the point of measurement. The mass is deduced from the analysis of the effect of coupling due to inertia; damping; and stiffness.

It is possible to measure very small rate of turn about an axis perpendicular to the plane of vibration, by means of the phase relationship between the induced and exited vibrations.

The phase relationship, unlike the amplitude ratio, is independent of the damping present in the system. Hence the sensitivity of the instrument is not affected by the increase in damping.

This system measures small rates of turn by a mass-spring-damper system under some simplified assumptions. One of these is that the excitation acting on the vibratory system is harmonic. This assumption, however, can lead to the system response directly by assuming a harmonic solution. This paper is devoted to study the capability and the efficiency of the mass-spring-damper system to measure rates of turn when excited by small random forces in addition to the harmonic forces.

Equations of motion and system description:

The system is in the figure. It consists of a point mass \( m \) constrained to move in the plane \( OXY \) of a rectangular set of axes \( OXYZ \), which is rotating in space at an angular velocity \( \Omega \) about a non-accelerating origin \( O \).
When the mass is displaced from equilibrium a distance \( r \) where \( r \) is measured from the origin \( O \), the mass will be subjected to the forces \( F_x \) and \( F_y \) in the \( OX \) and \( OY \) directions respectively due to the effects of stiffness, damping, inertia and the coupling terms.

The forces in relation to the displacement, velocity and acceleration are expressed as:

\[
F_x = C_1 \dddot{x} - K_1 \dddot{x} + C_i \dddot{y} + C_d \ddot{y} + C_s y \\
F_y = C_2 \dddot{y} + K_2 \dddot{y} + C_i \dddot{x} + C_d \ddot{x} + C_s x
\]  

(1)

Where

- \( C_1 \) and \( C_2 \) are the viscous damping coefficients in the \( OX \) and \( OY \) directions respectively;
- \( K_1 \) and \( K_2 \) are the spring constants in \( OX \) and \( OY \) directions respectively;
- \( C_i, C_d, \) and \( C_s \) are inertia, damping, and stiffness coupling coefficients; and are assumed equal in both \( OX \) and \( OY \) directions.

The exciting forces which act on the mass are harmonic forces in the \( OX \) and \( OY \) directions. The forces are of the same frequency \( \omega \), but have a phase difference \( \psi \) between them.
The equations of motion for the system are obtained by applying Newton's second law as follows:

\[ m \ddot{a}_1 = P_x - F_x \]

\[ m \ddot{a}_2 = P_y - F_y \]  \hspace{1cm} (3)

Where

\( a_1 \) and \( a_2 \) are the absolute accelerations of the system in the OX and OY directions.

The absolute accelerations of the mass, in general, can be written in a vector form as:

\[ \ddot{a} = \frac{\partial^2 \ddot{r}}{\partial t^2} + \frac{\partial \dot{\Omega}}{\partial t} \times \dot{r} + \ddot{\Omega} \times (\dot{\Omega} \times \dot{r}) + 2 \dot{\Omega} \times \frac{\partial r}{\partial t} \]  \hspace{1cm} (4)

Where

\[ \ddot{\Omega} = \Omega_1 \dddot{i} + \Omega_2 \dddot{j} + \Omega_3 \dddot{k} \]  \hspace{1cm} (5)

\[ \ddot{r} = x \dddot{i} + y \dddot{j} \]

(Since the mass is constrained to move in the OXY plane, then the displacement in the Z-direction equals zero)

Thus, equations (1) are reformulated by the derivation (3) as follows:

\[ \ddot{x} + 2 \Omega_1 \omega_{n1} \ddot{x} + \omega_{n1}^2 x + u_1 \dddot{y} + (\omega_{n1} u_{dl} - 2 \Omega ) \omega_{n1} \ddot{y} \]

\[ + u_{s1} \omega_{n1}^2 y = \omega_{n1}^2 x_s e^{j \omega t} \]

Similarly, the second equation are:

\[ \ddot{y} + 2 \Omega_2 \omega_{n2} \ddot{y} + \omega_{n2}^2 y + u_1 \dddot{x} + (\omega_{n2} u_{dl} + 2 \Omega ) \omega_{n2} \ddot{x} \]

\[ + u_{s2} \omega_{n2}^2 y = \omega_{n2}^2 y_s e^{j (\omega t + \psi)} \]  \hspace{1cm} (6)

Where \( \omega_{n1} \) and \( \omega_{n2} \) are undamped natural frequencies in OY and OX directions.
The effect of random forces is studied by adding a small random force term to the harmonic force term. These random forcing functions are expressed as Gaussian stochastic processes.

When small random force terms are added, the equations of motion becomes as follows:

\[
\begin{align*}
\ddot{x} + 2\xi_1 \omega_{n1} \dot{x} + \omega_{n1}^2 x + u_i \dot{y} - (\omega_{n1}^2 u_{d1} - 2\Omega) \dot{y} \\
u_s \omega_{n1}^2 y &= \omega_{n1}^2 (x_s e^{j\omega t} + \xi \mathcal{W}_1(t)) \\
\ddot{y} + 2\xi_2 \omega_{n2} \dot{y} + \omega_{n2}^2 y + u_i \dot{x} + (\omega_{n2}^2 u_{d2} + 2\Omega) \dot{x} \\
u_s \omega_{n2}^2 x &= \omega_{n2}^2 (y_s e^{j(\omega t + \gamma)} + \xi \mathcal{W}_2(t))
\end{align*}
\]

Where; \( \mathcal{W}_1(t) \) and \( \mathcal{W}_2(t) \) are Gaussian or Normal stochastic processes with zero mean and spectral density \( \sigma^2 \).

By using the state space method, these two equations are rewritten as follow:

\[
\begin{align*}
\dot{z}_R(t) &= Fz_R(t) + G u_R(t) \\
\dot{z}_I(t) &= Fz_I(t) + G u_I(t)
\end{align*}
\]

Where subscripte \( R \) and \( I \) refer to the real and the imaginary parts.
Where,

\[
P = \frac{1}{1 - u_i^2} \begin{bmatrix}
0 & 1 - u_i^2 \\
-\omega_{n1}^2 + \omega_{n2}^2 u_{s2} u_i & -2\frac{1}{2} \omega_{n1} + u_i(\omega_{n2} u_{d2} + 2\Omega) \\
0 & 0 \\
\omega_{n1}^2 u_i - \omega_{n2}^2 u_{s2} & 2\frac{1}{2} \omega_{n1} u_i - u_i(\omega_{n2} u_{d2} + 2\Omega)
\end{bmatrix}
\]

\[
0
-\omega_{n1}^2 u_{s1} + \omega_{n2}^2 u_i
0
-\omega_{n1} u_{s1} u_i - \omega_{n2}^2
\]

\[
G = \frac{1}{1 - u_i^2} \begin{bmatrix}
0 & 0 \\
\omega_{n1}^2 & -\omega_{n2}^2 u_i \\
0 & 0 \\
-\omega_{n1} u_i & \omega_{n2}^2
\end{bmatrix}
\]

\[
Z_R^T(t) = \begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{bmatrix}
\]

\[
Z_I^T(t) = \begin{bmatrix}
y_1 \\
y_2 \\
y_3 \\
y_4
\end{bmatrix}
\]

\[
u_R^T(t) = X_s \cos \omega t + \epsilon \bar{w}_1(t) \\
y_s \cos(\omega t + \psi) + \epsilon \bar{w}_2(t)
\]

\[
u_I^T(t) = X_s \sin \omega t + \epsilon \bar{w}_1(t) \\
y_s \sin(\omega t + \psi) + \epsilon \bar{w}_2(t)
\]
Simulation:

The numerical method can be efficiently used in solving the system of equations. A case study must be considered for solving the system equations numerically. The performance of the system at any case will be estimated after we know the performance of this case study.

A case study:

The parameters of the system are selected as follows:

Frequencies:

The natural frequency in OY direction ($\omega_{n2}$) must be equal to the frequency of the exciting forces ($\omega$) which is very important to be at resonance. The amplitude of ($Y$) is maximum at resonance and the $\text{arg} (Y/X)$ not affected by the change of the damper characteristic.

The natural frequency in OX direction ($\omega_{n1}$) can have any value which will not affect the system solution. The natural frequency $\omega_{n1}$ must not be equal to the frequency of the exciting forces ($\omega$).

The assumed values of the frequencies are as follows:

\[
\begin{align*}
\omega_{n1} &= 3 \text{ sec}^{-1} \\
\omega_{n2} &= 2 \text{ sec}^{-1} \\
\omega &= 2 \text{ sec}^{-1}
\end{align*}
\]

Damping factors: ($\xi_1$ & $\xi_2$)

The damping factors in OX and OY directions are assumed equal to 0.1. The change in this values does not change the phase angle.
Coupling terms : - \(( u_i, u_s, & u_d )\)

The system responses are obtained at different values of coupling terms.

Results

Using the computer program, the responses of the system are obtained with respect to the rates of turn. Some of the obtained results will be discussed as follows.

The Effect of Inertia coupling : -

The coupling terms \( u_i = 0.01 \) and the rest equal zero inserted into equations \(( 7 )\) we obtain :-

\[
\ddot{x} + 2j \omega_1 \omega_1 \dot{x} + \omega_1^2 x + u_i \dot{y} - 2\omega \dot{y} = \omega_n^2 ( X e^{j \omega t} ) + \epsilon \left( \nu_1(t) \right)
\]

\[
\ddot{y} + 2j \omega_2 \omega_2 \dot{y} + \omega_2^2 y + u_i \ddot{x} + 2\omega \ddot{x} = \omega_n^2 ( Y e^{j \omega t} ) + \epsilon \left( \nu_2(t) \right)
\]

Figure ( 1 ) shows plots of the magnitude \(| Y/X |\) versus \( \omega \) with some selected values of \( \epsilon \). From figure ( 1 ), note the error in measuring rates of turn. The symmetry about the vertical line \( \omega = 0 \) is lost.

Figure ( 3 ) shows plots of the percentage error in the magnitude. We note that the error percent in measuring the magnitude of the rates of turn is increased while the para-
The error percent when $\Omega$ positive is less than when $\Omega$ negative.

Figure (2) shows plots of the arg $(Y/X)$ versus $\Omega$ with the same selected values of $\varepsilon$. Figure (4) shows plots of the error percent in the phase angle shift between $Y$ and $X$. This figure shows the percentage error in the direction of the small rates of turn. Note that the error percent is increased while the randomness increased.

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**The Effect of Stiffness coupling:**

Introducing the coupling terms $u_{s1}$ and $u_{s2}$ not equal zero and the rest equal zeros into equations (1) we obtain:

$$
x + 2j_1 \omega_{n1} x + \omega^2_{n1} x - 2\Omega y + \omega^2_{n1} u_{s1} y = \omega^2_{n1} \left( x_s e^{j \omega t} + \varepsilon W_1(t) \right)
$$

$$
y + 2j_2 \omega_{n2} y + \omega^2_{n2} y + 2\Omega x + \omega^2_{n2} u_{s2} x = \omega^2_{n2} \left( y_s e^{j \omega t} + \varepsilon W_2(t) \right)
$$

Typical plots of the magnitude of the solution $|Y/X|$ versus $\Omega$ for $u_{s1}$ and $u_{s2}$ equal 0.01 and various values of $\varepsilon$ are shown in figure (5). Note that the symmetry about the vertical line $\Omega = 0$ is lost and there are certain error in measuring the magnitude of the small rates of turn.
The error percent in measuring the magnitude and directions of the small rates of turn are shown by the two figures (7), and (8). Figure (7) shows plots of the two curves of the percentage error related to the parameter equal 0.1, and 0.2. We note that the error percent are increased while randomness increased. Figure (8) shows plots of the error percent in the direction of the small rates of turn for the same selected values of $\varepsilon$ as the previous amplitude curves, which permit the observation that the percentage error have negative values up to $\Omega$ near to zero and then have positive values.

Conclusions:

The percentage of error of the measurements of the mass-spring-damper system with harmonic forces and small random forces does not exceed .15 percent, when the random forces are equal 20 percent of the harmonic forces. It has been recommended to be used in measuring both small and very small rates of turn.

REFERENCES

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Fig. 1  Variation of $|Y/X|$ versus $\Omega$ for $r_2 = 1$, $s = 1$, $u_1 = 0.1$, $u_{a1,2} = u_{d1,2} = 0$, and various values of $\epsilon$.

Fig. 3  Variation of the percentage error in magnitude versus $\Omega$ for $\epsilon = 0.1, 0.2$. 

Fig. 2 Variation of \( \text{arg}(Y/X) \) versus \( \Omega \) for \( r_2 = 1 \),
\( u_i = 0.01 \), \( u_{1,2} = 0 \), and various values of \( \epsilon \).

Fig. 4 Variation of the percentage error in direction versus \( \Omega \) for \( \epsilon = 0.1, 0.2 \).
Fig. 5 Variation of $|Y/X|$ versus $\Omega$ for $r_2=1$, $\delta_l = 0.1$, $u_{s_1,2} = 0.01$, $u_{s_1} = u_{s_2,1} = 0$, and various values of $\epsilon$.

Fig. 7 Variation of percentage error in direction versus $\Omega$ for $\epsilon = 0.1, 0.2$. 
Fig. 6 Variation of $\arg(Y/X)$ versus $\Omega$ for $r_2=1$, $u_{g_1,2}=0.01, u_1 = u_{d_1,2}=0$, and various values of $\epsilon$.

Fig. 7 Variation of percentage error in amplitude versus $\Omega$ for $\epsilon = 0.1, 0.2$. 