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APPLICATION OF FINITE ELEMENT METHOD IN SOLVING THE VIBRATION PROBLEM OF CIRCULAR PLATES

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ABSTRACT

The finite element method is applied to the free transverse vibration of circular plate. The assumptions of the classical plate theory are applied in deriving the basic equations required for the solution of the problem. The triangular bending element used in describing the bending of the plate is characterized with only nine degrees of freedom. Consequently the transverse deformation of the element is expressed by a polynomial interpolation function of third order.

Using the area coordinate system, the expression for each of stiffness and mass matrices is derived in more convenient form. Simplification processes are used in the derivation of these matrices using a transformation matrix.

A complete modal analysis for the plate is obtained by a constructed computer program written with FORTRAN-IV language. Fortunately this triangular element gave good results with respect to those obtained by the exact method.

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KEYWORDS

Finite element method, transverse vibration, circular plate,triangular bending element, stiffness matrix,mass matrix, natural frequency,natural modes.

INTRODUCTION

The earliest consideration of the subject of transverse vibrations of circular plates was in the Ninteenth century by Poisson [1] , who analyzed radiallysymmetric, free transverse vibrations. Kirchhoff [2] , gave the general solution of the problem of free vibrations. From the middle of the twentieth century, the circular plates havebeen studied dynamically by Reismann[3] and others. Conway [4] , Gallego Juarze[5] and other authors have studied the vibration of circular plates with variable thickness.

Exact solutions are available for only few cases. In the case of plates with general thickness variation and mixed boundary conditions, a number of approximate methods is used. In recent years, the finite element method proved to be a powerful technique for the vibration analysis of the plates. This method was studied by authors such as Zienkiewicz[6], Kirkhope and Welson[7].

In this paper the finite element method is applied to solve the vibration problem of circular plates. A triangular bending element with only nine degrees of freedom is used to describe the bending of the plate. The expression for each of stiffness and mass matrices are derived in more convenient form. By using the computer programing, a complete modal analysis of the plate is presented to be valid for solution the plates with uniform or variable thicknesses. The application is carried on the plates with uniform thickness only. The application for the plate with variable thickness needs a computer with large capacity.

PROBLEM FORMULATION

The assumption of the classical plate theory will be used in the derivation of the solution. In our work, the triangular element will be used in order to describe the domain surface of the plate.

Triangular Plate Bending Element:

In the small-deflection theory of thin plates, the transverse(normal) deflection W is uncoupled from the in-plane deflection u and v in x and y coordinates respectively. The element used has three nodes , one at each corner. The displacements at a node have three components, w is the transverse deflection, θ_x is the rotation about x-axis and θ_y is the rotation about y-axis.

$$\lceil \delta_i \rceil = \lfloor W_i, \theta_{xi}, \theta_{yi} \rfloor^T = \lfloor w_i, (\frac{\partial w}{\partial y})_i, (\frac{\partial w}{\partial x})_i \rfloor^T \qquad (1)$$

Consequently the nodal displacements for the element may be written as:

$$\lceil \delta \rceil^{e} = \lfloor \delta_1 , \delta_2 , \delta_3 \rfloor^{T} \tag{2}$$

Shape Function

The shape function describes the distribution of the deformation assumed over the element. Therefore, we must choose of the pattern shape of the distribution to satisfy the continuity inside the element. The unknown deformation W can be expressed by a polynomial interpolation function of third order including 10 terms. Since only nine independent degrees of freedom are imposed, the full cubic expansion polynomial will be reduced to the following form in order to limit the number of unknowns to nine [8],

$$W=\alpha_1+\alpha_2 x +\alpha_3 y+\alpha_4 x^2 +\alpha_5 xy +\alpha_6 y^2+\alpha_7 x^3+\alpha_8 (x^2 y+xy^2)+\alpha_9 y^3 \tag{3}$$

It is perferable to express that series by another coordinate system called the area coordinate for the existance of integration equations which simplify the evaluation of area integrals required during further analysis. The displacement of the plate can be described in the form:

$$W = \lfloor Z \rfloor \lceil \xi \rceil \tag{4}$$

where

$$\lfloor Z \rfloor = \lfloor L_1, L_2, L_3, (L_1^2 L_2+ \phi) , (L_1^2 L_3+\phi), (L_2^2 L_1+\phi), (L_2^2 L_3+\phi), (L_3^2 L_1+\phi), (L_3^2 L_1+\phi) \rfloor \tag{5}$$

$$\lceil \xi \rceil = \lfloor \xi_1, \xi_2, \xi_3 , \xi_4, \xi_5 , \xi_6, \xi_7, \xi_8, \xi_9 \rfloor^{T} \tag{6}$$

$$\phi = 0.5 \, L_1 L_2 L_3 \tag{7}$$

In Cartisian coordinate system, each node is defined with x,y,coordinates, while in area coordinate each node is defined by L_1, L_2 and L_3 values. The relation between the two systems is given by:

$$\begin{bmatrix} L_1 \\ L_2 \\ L_3 \end{bmatrix} = \frac{1}{2\Delta} \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} 1 \\ x \\ y \end{bmatrix} \tag{8}$$

where Δ is the area of the element , and

$$a_i = x_j y_k - x_k y_j , \quad b_i = y_j - y_k , \quad c_i = x_k - x_j \tag{9}$$

The suffix i takes the value 1,2,3. The suffixes j and k permute in a cyclic order.

The nine unknown coefficients $\xi_1, \xi_2 \ldots \xi_9$ can be determined by substituting the values of the nodal displacement defined in eqn.(2) into eqn(4).

$$\lceil \delta \rceil^{e} = [v] \lceil \xi \rceil \qquad (10)$$

$$\lceil \xi \rceil = [v]^{-1} \lceil \delta \rceil^{e} \qquad (11)$$

where $[v]$ is 9 x 9 transformation matrix. Eqns.(4) and (11) are leading to :

$$W = \lfloor Z \rfloor [v]^{-1} \lceil \delta \rceil^{e} = \lfloor N \rfloor \lceil \delta \rceil^{e} \qquad (12)$$

where $\lfloor N \rfloor$ is the shape function. The final form of the shape function for i node can be written in the form:

$$\lfloor N \rfloor_i^T = \begin{Bmatrix} L_i + L_i^2 L_j + L_i^2 L_k - L_i L_j^2 - L_i L_k^2 \\ b_k (L_i^2 L_j + \phi) - b_j (L_K L_i^2 + \phi) \\ c_k (L_i^2 L_j + \phi) - c_j (L_k L_i^2 + \phi) \end{Bmatrix} \qquad (13)$$

where i=1,2,3 and i,j,k are corresponding to the nodes 1,2,3 in cyclic permutation of suffixes 1-2-3.

Strain-Displacement Relationship

The strain of the bending plate can be expressed in the matrix form:

$$\lceil \varepsilon \rceil = -z \lfloor \frac{\partial^2}{\partial x^2}, \frac{\partial^2}{\partial y^2}, 2\frac{\partial^2}{\partial x \partial y} \rfloor^T \lfloor N \rfloor^e \lceil \delta \rceil^e = z[B]^e \lceil \delta \rceil^e \qquad (14)$$

Then the strain displacement transformation matrix will be:

$$[B]^e = \lfloor \frac{\partial^2}{\partial x^2}, \frac{\partial^2}{\partial y^2}, 2\frac{\partial^2}{\partial x \partial y} \rfloor \lfloor N \rfloor^e \qquad (15)$$

Stress-Strain Relationship

The stress components of the bending plate are proportional to the values of the bending, twisting moments. These moments can be summarized in the matrix form:

$$\lfloor M_x, M_y, M_{xy} \rfloor^T = -[EL] \lfloor \frac{\partial^2}{\partial x^2}, \frac{\partial^2}{\partial y^2}, 2\frac{\partial^2}{\partial x \partial y} \rfloor^T \qquad (16)$$

The elements of the stress vector are related to the corresponding moment vector by the following relation:

$$\lceil \sigma \rceil = \frac{1}{w_b} \lceil M \rceil = \frac{12}{h^3} [EL] \lceil \varepsilon \rceil \qquad (17)$$

where

$$[EL] = D\begin{bmatrix} 1 & \nu & o \\ \nu & 1 & o \\ o & o & \frac{1-\nu}{2} \end{bmatrix} \qquad (18)$$

Derivation of the Element Stiffness Equation

The derivation of the element stiffness matrix is based on the principle of stationary potential energy. This potential energy is equal to the strain energy Λ besides to the potential energy W_p

$$\pi = \Lambda + W_p = \tfrac{1}{2} \int^{V} \{\varepsilon^e\}^T \{\sigma^e\} dv - \{\delta^e\}^T \{F^e\} \qquad (19)$$

where $[F]^e$ is the applied load vector, dv is a volume differential element. The final form of the element stiffness matrix will be

$$[K^e] = \int^{\Delta} [B^e]^T [EL] [B^e] d\Delta \qquad (20)$$

Simplification Processes

The derivation of the stiffness matrix corresponding to the nine nodal values of the triangular element requires a large effort. In this method, the nodal displacement vector will be partitioned into the deflections(w_1, w_2, w_3) and the stopes (θ_{xi}, θ_{yi}; i=1,2,3). If the nodes of the element are denoted at the unloaded position by $1_o, 2_o, 3_o$, the actual displacement w can be divided into two parts

$$W = W^R + W^* \qquad (21)$$

where W^R represents the rigid body displacement from the unloaded position to the deformed position 1,2,3 which can be expressed by

$$W^R = L\ L_1\ L_2\ L_3\ [W_1\ W_2\ W_3]^T \qquad (22)$$

and W^* represents the deflection relative to the plane through, 1,2,3 which is recognized as a deflection due to simply supported element,

$$[\delta_i^*] = [\theta_{xi}\ \ \theta_{yi}]^T \qquad (23)$$

where i = 1,2,3

From eqns.(21),(22) and (23), the relation between the relative and actual nodal displacements will be

$$\{\delta^*\} = [\delta_1\ \delta_2\ \delta_3]^{*T} = [T]\ \{\delta^e\} \qquad (24)$$

where [T] is the displacement transformation matrix, [9] and [10]

Reduced Element Stiffness Matrix

Since the reduced element has only the rotational nodal displacements, then the shape function $[N^e]$ 1 x 9 will also be reduced to become $[N^*]$, 1 x 6 , see [10] . Also the Matrix $[B^e]$ reduced to become $[B^*]$

Now the expression of the reduced stiffness matrix of simply supported plate element with constant thickness will be

$$[K^*] = \int^{\Delta} [B^*] \; [EL] \; [B^*] \, d\Delta \tag{25}$$

This expression is derived in the following convenient form:

$$K^*_{es} = \frac{D}{12 \times 192\Delta^3} \left\{ 2\sum_{p=1}^{3} K_{es}(p,p) + \sum_{p=1}^{3} K_{es}(p,q) \right. \tag{26}$$

where:

$$K_{es}(p,p) = B(1,e,p)B(1,s,p)+B(2,e,p)B(2,s,p)+L_{\phi}B(3,e,p)B(3,s,p) + \nu[B(1,e,p)\,B(2,s,p)+B(1,s,p)B(2,e,p)]$$

and

$$k_{es}(p,q) = B(1,e,p)B(1,s,q)+B(2,e,q)\,B(2,s,p)+B(1,e,q)B(1,s,p)+ B(2,e,q)B(2,s,p)+L_{\phi}[B(3,e,p)B(3,s,q)+B(3,e,q)B(3,s,p)] + \nu[B(1,e,p)B(2,s,q)+B(2,e,p)B(1,s,q)+B(1,e,q)B(2,s,p)+ B(2,e,q)B(1,s,p)]$$

and $q=p+1$ where $p=1,2$ and $q=1$ where $p=3$, $L_{\phi} = (1-\nu)/2$

The B( ) values are determined from the expression of $[B^*]$, see [10].

Now the actual stiffness matrix will be obtained from the reduced element stiffness matrix by using the principle of the virtual work:

$$[K^e] = [T]^T [K^*] [T] \tag{27}$$

Element Mass Matrix:

For our problem, we will use the procedure of consistent mass matrix which may be suitable especially with using the triangular element with nine degrees of freedom only.

The general expression of the reduced triangular element of constant thickness h, can be written as:

$$[M^*] = \rho h \int^{\Delta} \lfloor N^* \rfloor^T \lfloor N^* \rfloor \, d\Delta \tag{28}$$

The element mass matrix $[M^e]$ can be obtained by substituting the reduced element mass matrix [M*] in the following transformation expression.

$$[M^e] = [T]^T [M^*] [T] \tag{29}$$

EQUATION OF MOTION

For the free vibration, the motion of the lightly damped plate is presented by the matrix differential equation in the form

$$[M]\{\ddot{\delta}\} + [K]\{\delta\} = 0 \tag{30}$$

Where [M] and [K] are the overall mass and stiffness matrices respectively, $\{\delta\}$, $\{\ddot{\delta}\}$ are the overall vectors of nodal displacements and acceleration respectively.

Refering to equation of motion(30), where the free oscillations are harmonic, the displacement $\{\delta\}$ can be written as:

$$\{\delta\} = \{\bar{x}\} e^{j\omega t} \tag{31}$$

where $\{\bar{x}\}$ is a column matrix of the amplitude of the displacement $\{\delta\}$, ω is the natural frequency of oscillation and t is the time.

From eqns.(30) and (31), we can obtain the form of the eigenvalue problem as:

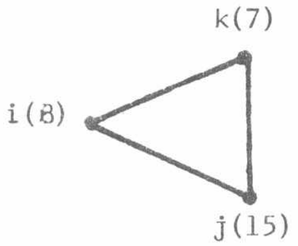
$$\left| [K^{-1}] [M] - \lambda [I] \right| \{\bar{x}\} = 0 \tag{32}$$

where $\lambda = 1/\omega^2$ is the eigenvalue and [I] is uni-matrix.

NUMERICAL RESULTS

The finite element method is applied on a uniform circular plate of 2.5 mm. thickness. The material of the plate is Pircspex with Young's modulus, $E = 262 \times 10^7$ N.M.$^{-2}$, Poisson's ratio ν=0.38 and a material density of 1.237×10^3 kg.m^{-3}. The plate is considered clamped at its boundary with outer radius of 150 mm. Fig.1 shows the plate mesh used in the computation of results, while table 1. shows the degrees of freedom corresponding to the nodals points of the plate.

The matrix iteration method is used in solving the eigenvalue problem, eqn.(32). Fig.2 shows the computer results of the natural frequencies and the mode shapes. The figure shows the deflections corresponding to the degrees of freedom. Fig.3 shows sketches for the two modes according to the results obtained.



No. of nodes,NN=17
No. of elements, NE)=24
No. of degree of freedom,M=27
Radius of the interior circle,3cm
Radius of the outer radius 15 cm

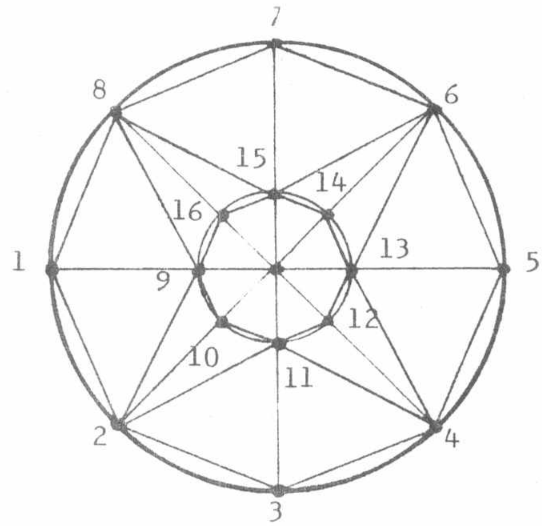


Fig. 1 Discertization of the Plate.

Table 1 The Degree of Freedom Corresponding to the Nodes.

Nodal Displ.	Degree of freedom corresponding to the node lable								
	Nodes from 1-8	9	10	11	12	13	14	15	16	17
W	0	1	4	7	10	13	16	19	22	25
θ_x	0	2	5	8	11	14	17	20	23	26
θ_y	0	3	6	9	12	15	18	21	24	27

Table 2. shows the results of the natural frequencies for the first two modes oo and 10 compared with those obtained by exact method.

Table 2. Natural Frequencies of Clamped Supported Circular Plate Using Finite Element Method.

Mode mn	Natural Frequency f, (Hz)		% Error
	Finite element method	Exact method	
00	88.54	81.83	8.19
10	185.45	170.64	8.6

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NATURAL FREQUENCY IN CYCLE/SEC
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Mode (00) 88.548195 185.455471 **Mode (10)**

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	Mode (00)		Mode (10)
1	1.0000	1	1.0000
2	-0.0000	2	-21.4791
3	21.6015	3	-61.6640
4	1.0219	4	0.0000
5	-10.8701	5	-48.8449
6	10.8701	6	-48.8449
7	1.0000	7	-1.0000
8	-21.6015	8	-61.6640
9	0.0000	9	-21.4791
10	1.0219	10	-0.5769
11	-10.8701	11	44.1790
12	-10.8701	12	44.1790
13	1.0000	13	-1.0000
14	0.0000	14	-21.4791
15	-21.6015	15	-61.6640
16	1.0219	16	-0.0000
17	10.8701	17	-48.8449
18	-10.8701	18	-48.8449
19	1.0000	19	1.0000
20	21.6015	20	-61.6640
21	-0.0000	21	-21.4791
22	1.0219	22	0.5769
23	10.8701	23	44.1790
24	10.8701	24	44.1790
25	1.2411	25	-0.0000
26	-0.0000	26	-25.0751
27	0.0000	27	-25.0751

Fig.2. Computer Results of Modes (00)and (10)

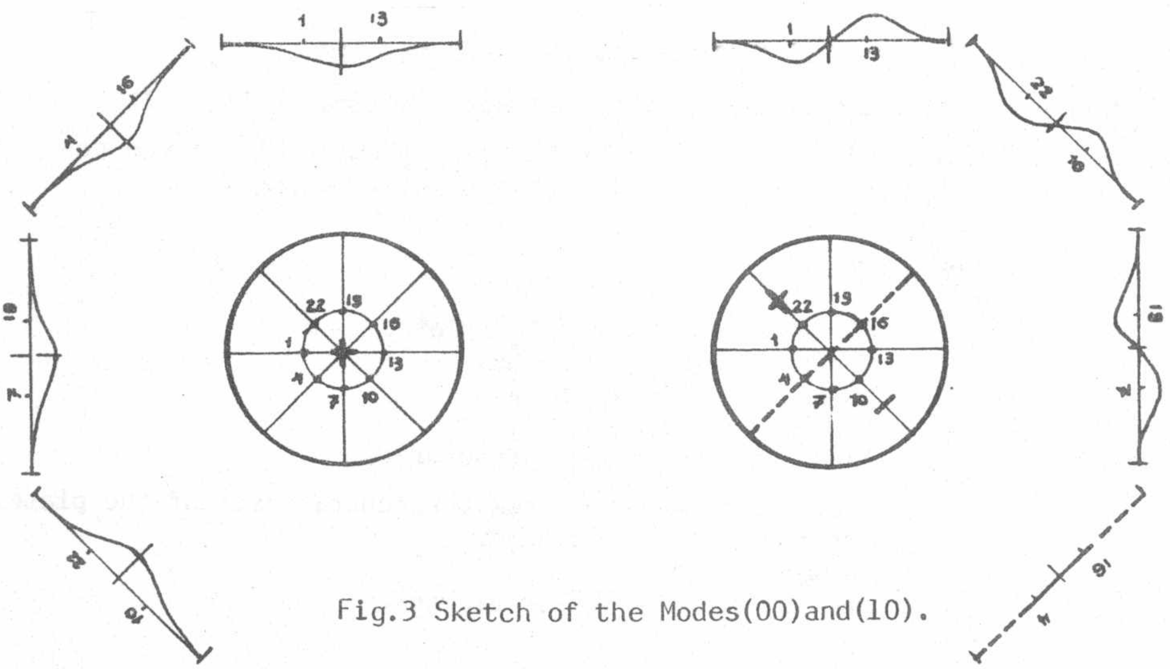

Fig.3 Sketch of the Modes(00)and(10).

CONCLUSIONS

The using of the simple triangular plate bending element gave good results while the number of the elements used is not enough. Also, the optimum grid of the plate required to give good results with minimum computation time was obtained by trial. It is prefered that the interior circle (of the plate mesh) to be near the centre of the plate than its boundary to minimize the computation time and to obtain a more than one mode. The results will be more accurate if the computer used in computation has a large capacity.

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NOMENCLATURE

D	Flexural rigidity of the plate.
f	Natural frequency, in Hz.
m	Number of nodal diameter.
n	Number of nodal circle.
w_b	Area Section modulus ofthe plate.
z	Transverse distance from the neutral axis of the plate.
ρ	Density of plate material.
ν	Poisson's ratio.
Δ	Area of the triangular element.