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STABILITY ANALYSIS OF AUTONOMOUS LINEAR SYSTEMS BY A

MATRIX DECOMPOSITION METHOD

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ABSTRACT

A special matrix decomposition method is presented for studying stability of autonomous linear systems. This decomposition appears to lead to a simple formulation capable of providing sufficient stability conditions for such systems. The method does not require solving the general state equations or searching for the appropriate Lyapunov functions. Several criteria regarding the stability of linear autonomous systems are established. A few examples of the applications of the results are also presented.

1. INTRODUCTION

The direct method of Lyapunov is a powerful tool for studying the stability properties. It has been used extensively in the past [1-17]. Unfortunately, finding the appropriate Lyapunov function is not always a simple task. Kalman and Bertram [2] introduced a method for finding the Lyapunov functions for continuous-time , free, linear, stationary dynamic systems. When the state variable is a n-vector, their method requires solving a set of $n(n+1)/2$ linear equations. Walker [3] introduced a technique to simplify Kalman's method for linear lumped-parameter systems. Moran [4], Mingori [5], and Walker and Schmitendoorf [6] investigated special cases of linear lumped-parameters systems. The well known Hurwitz's method (see [7]) requires the evaluation of the characteristic polynomials of the stability matrices, something we wish to avoid if the dimension of the stability matrix is large.

Recently, Lancaster and Rozsa [8] studied the general solution of Lyapunov equation which has a negative definite first time derivative. Barker, Berman, and Plemmons [9], Berman and hershkowitz [10], Kaszkurewicz and Hsu [11], and Khalil [12] studied the existance of a positive diagonal solution to this Lyapunov equation. Carlson, Datta and Johnson [13] investigated the asymptotic stability while the Lyapunov function has a negative semi-definite first derivative.

The purpose of this work is to introduce a simple method for investigating the stability properties of the autonomous linear systems including the lumped-parameter ones. This method is based on the properties of the matrix decomposition of the stability matrix of the given system. In the sequel, the matrix decomposition is first introduced. Stability of the equilibrium state of linear autonomous systems is considered and several stability criteria are established. Stability of lumped-mass structures is also studied and several examples are presented.

2. FORMULATION

A linear lumped-parameter structure is governed by

$$M\ddot{y}(t) + \hat{C}\dot{y}(t) + \hat{K}y(t) = 0 , \qquad (1)$$

where $y(t)$ is the response m-vector, and M, $\hat{C}$ and $\hat{K}$ are mxm real constant matrices corresponding to the mass, damping and stiffness of the structure, with M is assumed to be invertible.

Equation (1) may be restated as

$$\dot{x}(t) = A\,x(t) , \qquad (2)$$

where $x'(t)=[y' \ \dot{y}']$ is the response n-vector, n=2m, and A is a nxn real square matrix. The general formulation is presnted for any autonomous linear systems described by equation (2). Lumped-parameter structures will be treated as special examples.

Matrix Decomposition

Matrix A can, in general, be decomposed as

$$A = [W + L - R]D , \qquad (3)$$

where both $R=[r_{ii}]$ and $D=[d_{ii}]$ are diagonal matrices with D being a positive definite matrix, $W=[w_{ij}]$ is a skew-symmetic matrix, and $L=[l_{ij}]$ is such that $l_{ii}=0$ and $l_{ij}=0$ if $w_{ij}\neq 0$ for all i,j=1,2,..n. When L=0, the decomposition given by (3) becomes

$$A = [W - R]D , \qquad (4)$$

which is simple but useful form. Note that the matrix decomposition given by (3) or (4) is not unique.

The proposed matrix decomposition is based on the equivalent system method of the bond-graph theory [18]. In appendix A two examples are given to clarify the concept of these matrix decomposition and the corresponding bond-graph of the equivalent systems. It should be noted here that the matrix decomposition can be preformed without constructing the bond-graph of the equivalent system.

3. STABILITY ANALYSIS

In the following theorems let

$$O[N,F] = [N' \; F'N' \; \ldots . \; F'^{(n-1)}N']' , \tag{5}$$

where N is of order pxn and F is of order nxn, then O[N,F] is of order npxn. The matrix O[N,F] is called the observability matrix of the pair of matrices [N,F]. The pair [N,F] is observable if the matrix O[N,F] has full rank. i.e., rank[O[N,F]]=n (see [16]).

Lemma 1: Given any pair of matrices [N,F], then none of the eigenvectors of F are in the null space of N if [N,F] is observable.

Proof: Let λ be any eigenvalue of F and $x \neq 0$ is the associated eigenvector. Then it is follows that

$$O[N,F]\, x = \Omega N x , \tag{6}$$

where

$$\Omega = [I \; \lambda I \; \lambda^2 I \; \ldots \; \lambda^{(n-1)} I]' . \tag{7}$$

Since O[N,F] has full rank, then multiply (6) by $[O'O]^{-1}O'$ we get

$$x = [O'O]^{-1} \, O' \, \Omega \, N x \neq 0 . \tag{8}$$

Therefore, x is not in the null space of N.

Theorem 1: Consider any linear autonomous system (2) with A given by

(3). Then the equilibrium state, $x = 0$, is

a) Stable if $[2R-L-L']$ is a positive semi-definite matrix.

b) Unstable if $[2R-L-L']$ is a negative definite matrix.

c) Asymptotically stable if either

i) $[2R-L-L']$ is a positive definite matrix, or

ii) $[2R-L-L']$ is a positive semi-definite matrix and

$O[(R-L)D,A]$ has full rank.

Proof: Let

$$V = x'Dx. \tag{9}$$

Then

$$\dot{V} = x'[A'D+DA]x = - x'D[2R-L-L']Dx . \tag{10}$$

Therefore, $\dot{V} \leq 0$, $\dot{V} > 0$, or $\dot{V} < 0$ if $[2R-L-L']$ is a positive semi-definite, negative definite, or positive definite matrix respectively. These prove parts (a), (b), and (c, i) of the theorem. If $[2R-L-L']$ is a positive semi-definite matrix, then $\dot{V}$ given by (10) may be restasted as follows

$$\dot{V} = - x'[H+H']x \leq 0 . \tag{11}$$

where

$$H = D[R-L]D . \tag{12}$$

Now, if $O[[R-L]D,A]$ has full rank, then it may be shown that $O[H,A]$ has also full rank. Therefore, according to lemma 1, none of the eigenvectors of A are in the null space of H. Thus, $\dot{V}$ is not identically zero along any trajectory. This conclude the prove of part (c, ii).

4. LUMPED-PARAMETER SYSTEM

Stability of the lumped-parameter systems has attracted considerable attraction due to its significance in mechanical vibrations and structural dynamics. Several criteria regarding the stability of the equilibrium state were developed in [3-6]. Here, it will be shown that the present formulation contains these stability conditions as spacial cases.

Equation (1) may be rewritten as

$$\ddot{y} + C\dot{y} + Ky = 0 , \tag{13}$$

where

$$C = M^{-1}\hat{C} , \quad \text{and} \quad K = M^{-1}\hat{K}. \tag{14}$$

Equation (13) may be restated as equation (2) with

$$A = \begin{bmatrix} 0 & I \\ -k & -C \end{bmatrix} . \tag{15}$$

Matrix A can be decomposed into the forms given by (3) or (4). Theorem 1 is now applicable for studying the stability properties of the system. Using this theorem the following stability criteria for autonomous linear systems may now be established.

<u>Theorem 2:</u> Consider any linear lumped-parameter autonomous system (2) with A given by (15) such that K is a symmetric positive definite matrix. Then the equilibrium state, $x = 0$, is

a´) Stable if $[C+C']$ is a positive semi-definite matrix.

b´) Unstable if $[C+C']$ is a negative definite matrix.

c´) Asymptotically stable if either

i) $[C+C']$ is a positive definite matrix, or

ii) $[C+C']$ is a positive semi-definite matrix and $O[S,A)]$ has full rank, where

$$S = \begin{bmatrix} 0 & 0 \\ 0 & C \end{bmatrix}. \tag{16}$$

Proof: Let

$$K = T\Lambda T' \quad , \quad \hat{T} = \begin{bmatrix} T & 0 \\ 0 & T \end{bmatrix}, \tag{17}$$

where Λ is a diagonal matrix and T is an orthonormal matrix. Using equation (15), matrix U is introduced as

$$U = \hat{T}'A\hat{T} = \begin{bmatrix} 0 & I \\ -\Lambda & -T'CT \end{bmatrix} = [W - P]D , \tag{18}$$

where

$$W = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}, \; P = \begin{bmatrix} 0 & 0 \\ 0 & T'CT \end{bmatrix}, \text{and } D = \begin{bmatrix} \Lambda & 0 \\ 0 & I \end{bmatrix}. \tag{19}$$

Using V as given by (9), then

$$\dot{V} = - x'D[P + P']Dx. \tag{20}$$

Then $\dot{V} \leq 0$, or $\dot{V} > 0$ if $[C+C']$ is positive semi-definite, or negative definite matrix, respectively. This proves parts (a′) and (b′) of the theorem. Now, if $[C+C']$ is a positive definite matrix, then $\dot{V} \leq 0$, and

$$\text{rank} \begin{bmatrix} H \\ HU \end{bmatrix} = \text{rank} \begin{bmatrix} 0 & 0 \\ 0 & T'CT \\ 0 & 0 \\ -T'CT & -T'C^2T \end{bmatrix} = 2m = n. \tag{21}$$

where H as given by (12) be now becomes

$$H = P . \tag{22}$$

Thus $O[H,U]$ has full rank which proves part (c′, i) according to part (c, ii) of theorem 1. If $[C+C']$ is a positive semi-definite matrix, and $O[S,A]$ has full rank, then part (c, ii) of theorem 1 implies the prove of part (c′, ii).

Part (c′, i) of theorem 2 includes the asymptotic stability conditions of Kelvin-Tait-Chelave given in [15]. Part (c′, ii) contains Moran′s theorem 1 given in [4] and also the theorem presented by Walker and Schmitenderf in [6] as special cases.

5. APPLICATIONS AND EXAMPLES

several examples illustrating the applications of the developed theorems are presented in this section.

Example 1: Consider the system given in example A.1 in Appendix A. Since R is a positive definite matrix, then, in accord with part (c, i) of theorem 1 with $L = 0$, the equilibrium state is asymptotically stable. Furthermore, the appropriate Lyapunov function as obtained in [2] is given by (9).

Example 2: Consider the system given in example A.2 of Appendix A. The equilibrium state is asymptotically stable in according to part (c, i) of theorem 1 since $[2R-L-L']$ is a positive definite matrix. Equation (9) is also a Lyapunov function for this example.

Example 3: Consider the equation of motion of a single degree-of-freedom oscillator

$$\ddot{y} + 2\xi\dot{y} + \omega^2 y = 0 , \qquad (23)$$

where $\xi > 0$. Equation (23) may restated as equation (2) with $x = [y \quad y]$, and

$$A = \begin{bmatrix} 0 & 1 \\ -\omega^2 & -2\xi \end{bmatrix} = [W-R]D \;, \tag{24}$$

where

$$W = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \; D = \begin{bmatrix} \omega^2 & 0 \\ 0 & 1 \end{bmatrix}, \; \text{and } R = \begin{bmatrix} 0 & 0 \\ 0 & 2\xi \end{bmatrix} \tag{25}$$

It may be easily shown that the matrix O[2RD,A] as given by (5) has full rank. Part (c, ii) of theorem 1 implies that the equilibrium state, $x=0$, of system (23) is asymptotically stable. Note that using a Lyapunov function as given by (9), then $\dot{V}$ is not negative definite. However, part (c, ii) of theorem 1 implies asymptotic stability of the equilibrium state.

<u>Example 4</u>: Consider the system given by (2) with

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & -2 & 0 \\ 0 & -1 & 1 & -1 \end{bmatrix} = [W+L-R]D \;, \tag{26}$$

where

$$D=I, \; W=\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}, \; L=\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \; \text{and } R=\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{27}$$

Now, $[2R-L-L']$ is a positive semi-definite matrix , and O[(R-L)D,A] as given by (5) has full rank. Thus part (c, ii) of theorem 1 implies that the equilibrium state is asymptotically stable. This example can also be described as a lumped-parameter system given by equation (13) with

$$C = \begin{bmatrix} 2 & 0 \\ -1 & 1 \end{bmatrix}, \quad \text{and } K = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{28}$$

Then

$$[C+C'] = \begin{bmatrix} 4 & -1 \\ -1 & 2 \end{bmatrix}, \tag{29}$$

is a positive definite matrix. Now, part (c′, i) of theorem 2 leads to asymptotic stability of the equilibrium state. Applying the Routh-Hurwitz criteria [7], the same conclusion is reached after laborious manipulation. Using Kalman′s method [2] for constructing a Lyapunov function requires solving a system of ten linear equation which is also quite cumbersome.

<u>Example 5:</u> Consider a lumped-parameter system given by (13) with

$$C = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad \text{and } K = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}. \tag{30}$$

It may be easily shown that O[S,A] as given by (5), with A and S are given by (15) and (16), has full rank. Then, the equilibrium state is asymptotically stable in accord with part(c′, ii) of theorem 2. The same conclusion is reached by applying either Moran′s method [4] or Walker and Schmitendorf method [6]. Asymptotic stability of this system can be proved by the method given in [3], but solving a set of 6 linear equation is required.

6. CONCLUSION

A simple matrix decomposition technique capable of providing sufficient conditions for stability, asymptotic stability, and instability of the equilibrium state of autonomous linear systems including the lumped-parameter ones is presented. The bond-graph of the equivalent system is introduced to motivate the matrix decomposition procedure. However, the decomposition can be performed directly. The results are compared with the Lyapunov direct method as described by Kalman [2], Walker [3], and the Hurwitz technique [7]. It is observed that the developed stability criteria are much simpler to apply.

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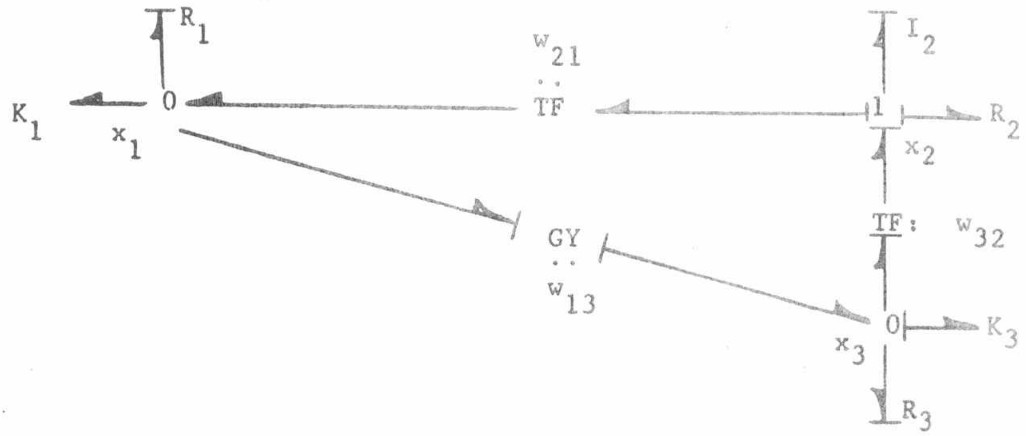
APPENDIX A

The concept of the matrix decomposition and the corresponding bond-graph of the equivalent systems is illustrated in the following two examples.

Example A.1: The equivalent system associated with

$$\dot{x}(t) = \begin{bmatrix} -3 & 4 & -2 \\ -2 & -2 & -2 \\ 1 & 2 & -1 \end{bmatrix} x(t) = A\, x(t) \qquad \text{(A.1)}$$

can be described by the following bond-graph:



Now, the general vector state equation of this bond-graph can be stated as

$$\dot{x}(t) = \begin{bmatrix} -R_1^{-1}K_1 & w_{21}I_2^{-1} & -w_{13}K_3 \\ -w_{21}K_1 & -R_2I_2^{-1} & w_{32}K_3 \\ w_{13}K_1 & -w_{32}I_2^{-1} & -R_3^{-1}K_3 \end{bmatrix} x(t) . \qquad \text{(A.2)}$$

Then, A can be decomposed as:

$$A = [W-R]D \ , \qquad \text{(A.3)}$$

where

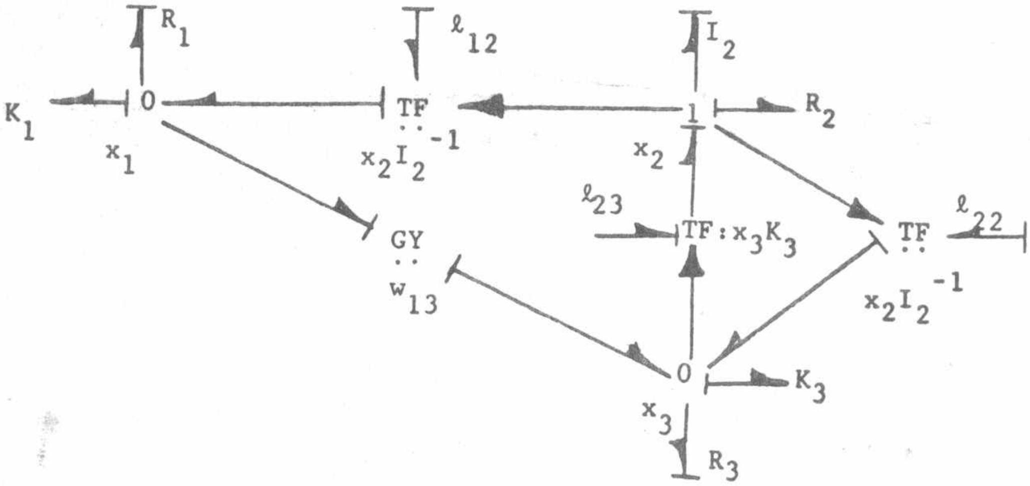
$$W = \begin{bmatrix} 0 & \frac{2}{3} & -\frac{1}{3} \\ -\frac{2}{3} & 0 & -\frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & 0 \end{bmatrix}, \quad R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{6} \end{bmatrix}, \quad D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix}. \qquad (A.4)$$

This example illustrates the decomposition of matrix A into the form given by (4).

Example A.2: Consider system equation (A.1) with

$$A = \begin{bmatrix} -1 & 4 & -2 \\ 0 & -4 & 2 \\ 1 & 2 & -2 \end{bmatrix} . \qquad (A.5)$$

Its equivalent system has the following bond-graph:



Its general equation of motion is given by

$$\dot{x}(t) = \begin{bmatrix} -R_1^{-1}K_1 & \ell_{12}I_2^{-1} & -w_{13}K_3 \\ 0 & -R_2I_2^{-1} & \ell_{23}K_3 \\ w_{13}K_1 & \ell_{32}I_2^{-1} & -R_3^{-1}K_3 \end{bmatrix} x(t) . \qquad (A.6)$$

Then, A can be decomposed into the general decomposition form given by (3), i.e.,

$$A = [W+L-R]D \quad , \qquad (A.7)$$

where

$$W = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad L = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$\text{and} \quad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}. \tag{A.8}$$