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THE RESTRICTIVE PADÉ APPROXIMATION FOR THE SOLUTION OF SINGULARLY PERTURBED PARABOLIC PDE

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ABSTRACT

This work is concerned with the numerical treatment of initial-boundary value problem for linear parabolic equations by a small parameter with the time derivative term. This problem is reduced to a stiff system of ODEs in time.The resulting system is solved by applying the restrictive Padé approximation of matrix exponentials. Numerical results are given and the method gives better results compared with the classical Padé approximation treatments.

KEY WORDS

Padé approximation, restrictive Padé approximation, exponential matrix, singularly perturbed equation, finite difference and parabolic partial differential equations

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1.INTRODUCTION

A restrictive Pade approximation technique is applied to many different problems. It has high accurate results in all cases. It was acting like an accelerating technique in some applications, and has other advantage features like satisfying very small absolute errors whatever the exact solution is too large in the model problem.[1],[2],[3].Here we use this method to approximate the solution of the singularly perturbed parabolic partial differential equation which arises in the modeling of various physical processes. The outline in the present paper is as follows. In Section 2, we present the statement of the problem and approximate u_{xx} by the usual three-point difference formula. By doing so the problem is reduced to a stiff system of ordinary differential equations (ODEs) in the time variable. In section 3, we define and explain the concept of restrictive Padé. In Section 4, the finite difference schemes approximating the solution of the model problem are derived. Numerical results and conclusion are presented in Section 5.

2.THE PROBLEM , DISCRETIZATION IN SPACE AND REDUCTION TO A SYSTEM OF ODES

We consider the singularly perturbed parabolic partial differential equation:

$$\delta u_t - k u_{xx} = f(x,t), \qquad 0<x<1 , \quad t>0 \qquad \text{(1-a)}$$

with initial condition:

$$u(x,0) = u_o(x), \qquad 0 \le x \le 1 \qquad \text{(1-b)}$$

and boundary condition:

$$u(0,t) = g_1(t), \qquad u(1,t) = g_2(t), \quad t>0 \qquad \text{(1-c)}$$

where $\delta>0$ is small, k is a given positive constant and $f(x,t)$ is a given heat source. By the usual three-point difference approximation to u_{xx},

$$u_{xx}(ih,t) = \frac{1}{h^2}[u((i+1)h,t) - 2u(ih,t) + u((i-1)h,t)] , \qquad i : 0(1)N , \quad (N+1)h=1,$$

the resulting semi-discrete approximation $U(ih,t)$ to $u(x,t)$ of (1) satisfies:

$$\frac{dU_i(t)}{dt} = \frac{k}{\delta h^2}[U_{i+1}(t) - 2U\ (t) + U_{i-1}(t)] + \frac{1}{\delta} f_i(t), \quad 1 \le i \le N, \qquad t>0; \qquad (2)$$

$$U_i(0) = u_o(ih), \quad U_o(t) = g_1(t), \quad U_{N+1}(t) = g_2(t).$$

This can be written in the matrix form:

$$\frac{dU(t)}{dt} + AU(t) = F(t), \quad U(0) = (U_1(0),...,U_N(0))^T \quad , \tag{3}$$

where *U(t)* is a column vector with N components :

$U(t) = (U_1(t),...,U_N(t))^{t'}, \quad U_i(t) = U(ih,t)$ and,

$$A = \frac{k}{\delta h^2}\begin{pmatrix} 2 & -1 & & & & \\ -1 & 2 & -1 & & 0 & \\ & \cdot & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & \cdot \\ & 0 & & -1 & 2 & -1 \\ & & & & -1 & 2 \end{pmatrix}, \quad F(t) = \begin{pmatrix} \frac{k}{\delta h^2} g_1(t) + \frac{1}{\delta} f(x_1,t) \\ \frac{1}{\delta} f(x_2,t) \\ \vdots \\ \frac{1}{\delta} f(x_{N-1},t) \\ \frac{k}{\delta h^2} g_N(t) + \frac{1}{\delta} f(x_N,t) \end{pmatrix}$$

The solution of this system of ordinary differential equations (3) is clear by using Varga [6] :

$$U(t) = \exp[-tA]U(0) + \int_0^t \exp[-(t-\tau)A]F(\tau)d\tau.$$

We now assume that the source heat f(x,t) and the boundary conditions are time independent and consequently the vector F(t) of (3) will be also time independent and the solution of (3) takes the simple vector form [5]:

$$U(t) = \exp[-tA]U(0) + [I - \exp(-tA)]A^{-1}F,$$

or equivalently,

$$\begin{aligned} U(t_o + \Delta t) &= \exp(-\Delta tA)U(t_o) + [I - \exp(-\Delta tA)]A^{-1}F \\ &= A^{-1}F + \exp(-\Delta tA)[U(t_0) - A^{-1}F]. \end{aligned}$$

Considering the Padé approximation of the order (n/m) to the exponential $\exp(-\Delta tA)$,we get this approximate solution to our system of equations (3).

$$U(t_o + \Delta t) = A^{-1}F + P_{e^{-\Delta t}}(n/m)[U(t_o) - A^{-1}F] \tag{4}$$

In the next section we consider the restrictive Padé approximation .

3.RESTRICTIVE PADE APPROXIMATION:

The restrictive Padé approximation of the function f(x) is a particular type of rational approximation , it can be written in the form :

$$RPA[M + \alpha / N]_{f(x)}(x) = \frac{\sum_{i=0}^{M} a_i x^i + \sum_{i=1}^{\alpha} \epsilon_i \, x^{M+i}}{1 + \sum_{i=1}^{N} b_i x^i}, \tag{5}$$

where the positive integer α does not exceed the degree of the numerator N , i.e. α=0(1)N,

$$f(x) - RPA[M+\alpha / N]_{f(x)}(x) = o(x^{M+N+1}). \tag{6}$$

the unknowns a_i, b_i and ε_i are to be determined.

Let f(x) have a Maclaurin series

$$f(x) = \sum_{i=0}^{\infty} c_i x^i , \tag{7}$$

from equations (5) - (7) we get:

$$\left(\sum_{i=0}^{\infty} c_i x^i\right)\left(1+\sum_{i=1}^{N} b_i x^i\right) - \sum_{i=0}^{M} a_i x^i - \sum_{i=1}^{\alpha} \epsilon_i x^{i+M} = o(x^{M+N+1}). \tag{8}$$

The vanishing of the first (M+N+1) powers of x on the left-hand side of (8) implies a system of (M+N+1) equations, and let

$$RPA[M+\alpha / N]_{f(x)}(x_j) = f(x_j) \quad , j = 1(1)\alpha,$$

imply a system of α equations. It means that the considered approximation is exact at the (α+1) Points $\{x_0 \equiv 0, x_1, x_2, \dots, x_\alpha\}$. Then the (M+N+$\alpha$+1) coefficients b_i , i= 1(1)N, a_i , i = 0(1)M and ε_i , i = 1(1) α can be derived from these systems.

4.DERIVATION OF THE FINITE DIFFERENCE SCHEMES USING THE RESTRICTIVE PADÉ APPROXIMATION OF THE EXPONENTIAL MATRIX:

4.1. By the same above concept and analogue argument, we can easily deduce that the restrictive Padé approximant of order [0/1] of the exponential matrix exp[-Δt A] has the form:

$$[RPA]_{e^{-\Delta t}}(0/1) = [I + (\Delta t + \varepsilon)A]^{-1},$$

using this approximation of exp[-ΔtA] instead of usual Padé approximation in equation (4) , the difference equation approximating the problem at the point (i,j) is the *i*th row of the equations:

$$[I + (\Delta t + \varepsilon)A]U^{j+1} = (\Delta t + \varepsilon)F + U^j \tag{9}$$

Taking for simplicity k=1 and g(t)=0 ,equation (9) can be written in the following matrix form:

$$\begin{bmatrix} 1+\frac{2r}{\delta}(1+\frac{\varepsilon}{\Delta t}) & -\frac{r}{\delta}(1+\frac{\varepsilon}{\Delta t}) & & & 0 \\ -\frac{r}{\delta}(1+\frac{\varepsilon}{\Delta t}) & 1+\frac{2r}{\delta}(1+\frac{\varepsilon}{\Delta t}) & -\frac{r}{\delta}(1+\frac{\varepsilon}{\Delta t}) & & \\ & \ddots & \ddots & \ddots & \\ & & -\frac{r}{\delta}(1+\frac{\varepsilon}{\Delta t}) & 1+\frac{2r}{\delta}(1+\frac{\varepsilon}{\Delta t}) & -\frac{r}{\delta}(1+\frac{\varepsilon}{\Delta t}) \\ 0 & & & -\frac{r}{\delta}(1+\frac{\varepsilon}{\Delta t}) & 1+\frac{2r}{\delta}(1+\frac{\varepsilon}{\Delta t}) \end{bmatrix} \begin{pmatrix} U_1 \\ U_2 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ U_N \end{pmatrix}^{j+1} =$$

$$\begin{pmatrix} \frac{\Delta t+\varepsilon}{\delta} f(x_1,t) \\ \frac{\Delta t+\varepsilon}{\delta} f(x_2,t) \\ \vdots \\ \frac{\Delta t+\varepsilon}{\delta} f(x_{N-1},t) \\ \frac{\Delta t+\varepsilon}{\delta} f(x_N,t) \end{pmatrix} + \begin{pmatrix} U_1 \\ U_2 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ U_N \end{pmatrix}^{j}.$$

This gives the equivalent scalar form:

$$-\frac{r}{\delta}(1+\frac{\varepsilon}{\Delta t})U_{i-1}^{j+1} + [1+\frac{2r}{\delta}(1+\frac{\varepsilon}{\Delta t})]U_i^{j+1} - \frac{r}{\delta}(1+\frac{\varepsilon}{\Delta t})U_{i+1}^{j+1} = \frac{\Delta t+\varepsilon}{\delta} f(x_i,t) + U_i^j \qquad (10\text{-}1)$$

4.2 . While approximating exp[- ΔtA] by the restrictive Padé of order **[1/1]** :

$$[\mathrm{RPA}]_{e^{-\Delta tA}}(1/1) = \frac{[I-(0.5+\varepsilon)\Delta tA]}{[I+(0.5-\varepsilon)\Delta tA]}.$$

yields the finite difference scheme:

$$\frac{(\varepsilon-0.5)r}{\delta}U_{i-1}^{j+1} + [1+\frac{(1-2\varepsilon)r}{\delta}U_i^{j+1} + \frac{(\varepsilon-0.5)r}{\delta}U_{i+1}^{j+1} =$$

$$\frac{(0.5+\varepsilon)r}{\delta}U_{i-1}^{j} + [1-\frac{(1+2\varepsilon)r}{\delta}]U_i^{j} + \frac{(0.5+\varepsilon)r}{\delta}U_{i+1}^{j} + \frac{\Delta t}{\delta}f(x_i) \qquad (10\text{-}2)$$

Putting $\varepsilon=0$ in (10-1),(10-2) , we turn back to the schemes arising of applying Padé approximants of orders (0/1),(1/1) respectively to the exponential matrix exp[- ΔtA]

Stability conditions

Using the matrix method, the eigenvalues of the amplification matrix of equation (10-1) are:

$$\lambda_s = \frac{1}{1+2\dfrac{(\Delta t+\varepsilon)}{\delta h^2}(1+\cos\dfrac{s\pi}{N+1})}, \qquad s,i=1(1)N\text{-}1,$$

then $\max\limits_s \ |\lambda_s| \le 1$ if $\varepsilon \ge -\Delta t$,

which gives the stability condition for the given algorithm.

For the scheme (10-2), the eigenvalues of the amplification matrix are:

$$\lambda_s = \frac{1-2r\dfrac{(0.5+\varepsilon)}{\delta}(1+\cos\dfrac{s\pi}{N+1})}{1+2r\dfrac{(0.5-\varepsilon)}{\delta}(1+\cos\dfrac{s\pi}{N+1})}, \qquad s,i=1(1)N\text{-}1,$$

the suggested method will be stable when $|\lambda_s| \le 1$ i.e. $\varepsilon \le \dfrac{\delta}{4r}$

5.NUMERICAL EXAMPLE

In this Section we present the results of a numerical experiment applying the schemes described in this paper to the following numerical example

$$\delta u_t - u_{xx} = 1-3x, \qquad (x,t)\in(0,1)\times(0,1) \tag{11-1}$$

with initial condition:

$$u(x,0) = \sin(\pi x) - .5x^2(1-x), \qquad x\in[0,1) \tag{11-2}$$

and boundary condition:

$$u(0,t) = u(1,t) = 0, \qquad t\in(0,1] \tag{11-3}$$

which has the exact solution:

$$u(x,t) = -0.5x^2(1-x) + \exp[-t\pi^2/\delta]\sin(\pi x).$$

Taking $N=9$ i.e. $h=0.1$, we calculate the solution at the time $t=10^{-5}$ studying two cases ,the first if the exact solution is known at the first level and the second without knowing the exact solution, we estimate a highly accurate method to know the solution at the first level, as following:

The first case: We apply the implicit methods (10-1) and (10-2) on the problem (11), the unknown parameter ε is to be determined, then we must know an additional condition ,u (x,k) to be given, i.e. $U(x,\Delta t) = u(x,\Delta t)$,then ε is given such that the truncation error for certain value of r is equal zero. Then we use this value in the finite difference scheme for approximating the solution in the other levels . In table (1) the absolute value of the errors are given a middle point x=0.5 for different choices of δ and Δt .These results also are compared with the results obtained by applying the classical Padé approximation method of order (0/1) in table (1) and of order (1/1) in table (2)

The second case: In this case we use Crank-Niclson method to find the solution at the first level with taking a smaller mesh width h1=0.001 and by considering 100 steps to this first level , thus the result is given by a higher accurate method . Then we are proceeding as in the first case. We get the results in Tables (3) and (4) for RP[0/1] and RP[1/1] respectively .

CONCLUSION

It can be seen that:

1. the absolute error in the restrictive Padé method is smaller than that arising when use the usual Padé approximation of order (0/1).
2. whence the attained solution was unbounded for Padé approximation of order (1/1) (see also Shamardan[4]), the restrictive Padé of the same order still has bounded accurate results. This illustrates another advantage of the considered method.
3. Both of restrictive Padé Approximants of orders (0/1) and (1/1) give almost the same results ,i.e diagonal and nondiagonal orders are of the same accuracy.
4. We note that smaller perturbation parameter gives less accurate results in all cases.

Table 1. The absolute value of the errors obtained by bothmethods Padé and restrictive Padé of order (0/1) at thepoint x=0.5 (The first case).

Perturbed Parameter δ	The time step Δt	Number of Steps J	Errors of P[0/1]	Errors of RP[0/1]
0.1	5×10^{-8}	200	8×10^{-6}	3×10^{-15}
0.01	2×10^{-8}	500	8×10^{-5}	8×10^{-14}
0.001	10^{-8}	1000	3×10^{-4}	4×10^{-14}
0.0001	5×10^{-9}	2000	3×10^{-3}	7×10^{-14}

Table 2. The absolute value of the errors obtained by both methods Padé and restrictive Padé of order (1/1) at the point x=0.5 (The first case).

Perturbed Parameter δ	The time step Δt	Number of Steps J	Errors of P[1/1]	Errors of RP[1/1]
0.1	5×10^{-8}	200	10^{-2}	4×10^{-14}
0.01	2×10^{-8}	500	10^{-1}	7×10^{-14}
0.001	10^{-8}	1000	0.6	8×10^{-14}
0.0001	5×10^{-9}	2000	0.3	7×10^{-14}

Table 3. The absolute value of the errors obtained by both methods Padé and restrictive Padé of order (0/1) at the point x=0.5 (The second case).

Perturbed Parameter δ	The time step Δt	Number of Steps J	Errors of P[0/1]	Errors of RP[0/1]
0.1	5×10^{-8}	200	8×10^{-6}	8×10^{-8}
0.01	2×10^{-8}	500	8×10^{-5}	8×10^{-7}
0.001	10^{-8}	1000	3×10^{-4}	7×10^{-6}
0.0001	5×10^{-9}	2000	3×10^{-3}	3×10^{-5}

Table 4. The absolute value of the errors obtained by both methods Padé and Restrictive Padé of order (1/1) at the point x=0.5 (The second case).

Perturbed Parameter δ	The time step Δt	Number of Steps J	Errors of P[1/1]	Errors of RP[1/1]
0.1	5×10^{-8}	200	10^{-2}	8×10^{-8}
0.01	2×10^{-8}	500	10^{-1}	8×10^{-7}
0.001	10^{-8}	1000	0.6	7×10^{-6}
0.0001	5×10^{-9}	2000	0.3	3×10^{-5}

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