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Stability Optimization of Thin–Walled Functionally Graded Beams

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Abstract: This paper presents an optimization model for improving stability levels of thin–walled composite beams under axial compressive loading. Optimum designs are obtained by maximizing the critical buckling load while maintaining the total structural mass at a prescribed value equals to that of a baseline design. The dual problem of minimizing the total structural mass under preserved buckling load is also addressed. The developed optimization models deal with slender beam–columns that are axially graded in both material and wall thickness. The main structure is constructed from uniform segments that are fabricated from a composite with different volume fractions of the constituent materials, making the physical and mechanical properties change piecewisly in the axial direction. Design variables include the volume fraction of the constituent materials, the wall thickness as well as the length of each segment composing the beam. The buckling load analysis is performed via finite element method, using a beam element with two degrees of freedom at each node. The resulting optimization problem has been formulated as a nonlinear mathematical programming problem solved by invoking the Matlab optimization toolbox routines, which implement the method of sequential quadratic programming interacting with the associated eigenvalue problem routine. The proposed mathematical models have shown that the use of material grading concept can be promising in raising stability boundaries without mass penalty and producing economical designs having enhanced stability as compared with their corresponding baseline designs. Finally, the given approach can be beneficial to guide structural engineers for choosing the significant design variables in proper and efficient way without violating economic feasibility requirements.

Keywords: Structure optimization, Column's buckling, Material grading, Stability and Finite element.

1. Introduction

Light weight, thin-walled structures loaded by compression may fail due to buckling [1,2]. Among them, axially compressed elastic columns play an important role, because of their wide application as real structures in aerospace, mechanical, marine and civil engineering. Increasing the stability of an elastic column against buckling usually requires increasing structural weight, which in turn, might violate the economic feasibility of the final design. Therefore, it is the role of structural engineers to raise the overall stability level through maximization of the critical buckling load, while maintaining the total structural mass at a specified value. A large number of publications have appeared on this topic where the Eigenvalue optimization algorithms were applied to either continuous or discretized finite

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element structural models. Keller [3] determined the strongest simply supported column having the maximum buckling load 33.33% higher than that of the uniform column. However, the obtained shapes with a highly non-linear geometries and zero cross sections at the support locations were generally impractical. Following Keller's work, several researchers investigated the strongest columns with other different boundary conditions using energy approach for continuous structural models [4], or finite element method for discretised models [5].

Hornbuckle and Boykin [6] considered cantilevers with circular cross section, where the associated buckling optimization problem was handled by Pontryagin's maximum principle resulting in a critical buckling load 11.5% higher than that of a uniform cantilever. Another optimization algorithm refered to as the constructive algorithm, was proposed by Ishida and Sugiyama [7] and applied to a finite element column model. Manicharajah et al. [8] also applied the finite element method in conjunction with an iterative procedure to optimize columns and plane frames against buckling under mass equality constraint. A local modification of each element was assessed by gradually shifting the material from the strongest part of the structure to the weakest one under the same cross sectional constraint $I = \alpha A^2$, where α is a constant.

A piecewise model concept was introduced by Maalawi [9] to fined optimized designs of elastic column having the maximum buckling stability limits. The associated optimization variables included the cross sectional dimensions as well as the length of each segment composing the column. Several solutions were given for both solid and tubular cross sections, where it was concluded that the use of piecewise models in structural optimization gives excellent results and can be promising for similar applications. A more detailed work by Maalawi and El-Chazly [10,11] considered stability optimization and practical shapes of the strongest columns built of uniform beam elements. Another trend for improving structural dynamic and stability characteristics employs the concept of material grading [12], in which the physical and mechanical properties vary spatially within the structure being optimized.

The Functionally Graded Material " FGM" concept was originated in Japan in 1984 during the space project, in the form of proposed thermal barrier material capable of withstanding high temperature gradients. Considering structural stability, Elishakoff and Rollot [13] presented closed-form solutions of the buckling load of a variable stiffness column. In addition, Elishakoff and Endres [14] considered closed-form solution for the mode shape and buckling load of an axially graded cantilevered column. Shi-Rongli and Batra [15] studied buckling of simply supported three-layer circular cylindrical shell under axial compressive load. The middle layer sandwhiched by two isotropic layers was made of an isotropic FGM with Young's modulus varied parabolically in the thickness direction. Classical shell theory was implemented under the assumption of very small thickness/radius and very large length/radius ratios. Numerical result showed that the buckling load increases with the increase in the average value of Young's modulus of the middle layer.

In the field of aeroelastic stability, Librescu and Maalawi [16] introduced the underlying concepts of using material grading in optimizing subsonic wings against torsional instability. They developed exact mathematical models allowing the material physical and mechanical properties to change in the wing spanwise direction, where both continuous and piecewise structural models were successfully implemented. Turner and Plaut [17] considered clamped-clamped columns using an iterative procedure based on the optimality criterion accomplished by the finite element method. The column was divided into 20 uniform elements with equal lengths, and the resulting optimization gain was 27.6% under the same quadratic constraint imposed on the cross sectional properties. They implemented SQ (sequentional quadratic programmer) algorithm to find the optimum value of the critical buckling load, they concluded that the volume fraction needs to be varied in the longitudinal direction.

The major goal of the present study is to apply the concept of axial grading, either in material properties, or wall thickness of the cross section, aiming at the achievement of the maximum possible stability limits without the penalty of increasing structural mass. The column's structural model is made of piecewise uniform segments, each of them has different length, and wall thickness and volume fraction of the constituent materials. Substantial improvement in the overall stability level has been proposed to prove the usefulness of the developed mathematical model in arriving at stiff column designs for a variety of configurations having arbitrary cross sectional shape.

2. Finite Element Formulation

Figure 1 shows an elastic, slender column structure of total length (L) constructed from any arbitrary number of uniform segments (N_s), each of which has either a different property of the material of construction or a different wall thickness (h) of the cross section. Such a configuration results in a piecewise axial grading of either the material of construction or the wall thickness of the cross section in the direction of the column axis. Each segment is further subdivided into a reasonable number of finite elements as shown in the Fig. 1. Before determining the exact critical buckling load P_{cr} and performing the necessary mathematics, it is important to bear in mind that design optimization is only as meaningful as its structural analysis model. Any deficiencies therein will certainly be reflected in the optimization process.

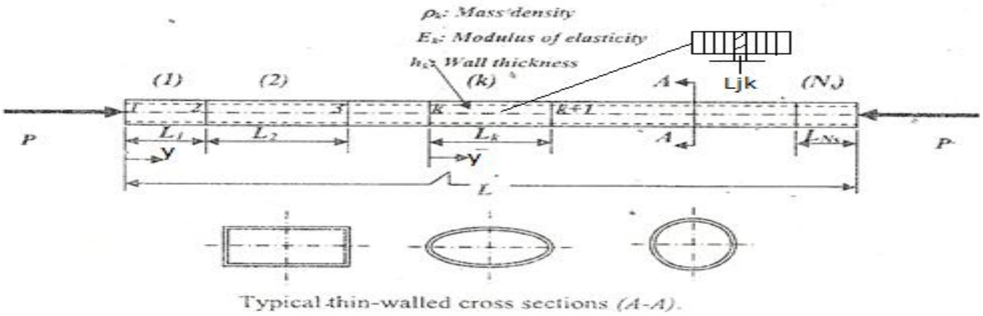

Fig. 1 General configuration of a piecewise axially graded column.

The various parameters are defined as follows in Table 1:

N_s = Number of segment.

$N_{E,k}$ = Number of finite elements in the k-th segment, k=1,2,…..N_s.

$\hat{L}_k$ = (L_k/L) = Normalized length of the k-th segment.

$\hat{L}_{jk}$ = (L_{jk}/L) = Normalized length of the j-th finite element in the k-th segment, j=1,2,…..N_E.

By applying the principle of stationary total potential energy, we need to evaluate the strain energy U_i, the external work W_{ex}, and the total potential energy Π of the column . For an elastic system, the work done by the external forces is stored as strain energy within the system. The bending strain energy for an element ($L_e = L_{jk}$) can be written as [5]:

$$U_i = \frac{1}{2}\int_0^{L_e} E_e\, I_e \left(\frac{d^2W}{dy_e^2}\right)^2 . dy_e \qquad (1)$$

where:

W(y) …. The bending displacement along the beam column.

$(E\,I)_e$ …. Bending stiffness of the element.

For the centrally loaded column, the external work done by edge loading can be written as:

$$W_{ex} = \frac{1}{2}\int_0^{L_e} P\,.\left(\frac{dW}{dy_e}\right)^2 .\,d\,y_e \tag{2}$$

where:

P …. Applied axial force.

The sum of the strain energy and potential energy of the system is the total potential energy. Using the symbol Π to denote the total potential energy of a system then:

$$\Pi = U_i - W_{ex} \tag{3}$$

Substituting from equations (1) and (2) we get:

$$\Pi = \frac{1}{2}\int_0^{L_e} E_e\, I_e \left(\frac{d^2W}{dy_e^2}\right)^2 .\,dy_e - \frac{1}{2}\int_0^{L_e} P\,.\left(\frac{dW}{dy_e}\right)^2 .\,d\,y_e \tag{4}$$

For equilibrium, the first variation of total potential energy must vanish:

$$\delta\Pi = 0 \tag{5}$$

The last expression is the mathematical statement of the principle of stationary total potential energy. A stationary value may correspond to a minimum or a maximum value of the total potential energy. A minimum value indicates that the equilibrium is stable, and a maximum value indicates that the equilibrium is unstable. Expressing a cubic displacement function in terms of the nodal displacements of the finite beam element, as shown in Fig. 2.

$$W(y_e) = [N_1 \quad N_2 \quad N_3 \quad N_4]\begin{bmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \end{bmatrix} \tag{6}$$

where, N_i, (i=1,2,3,4) are called the shape functions defined as follows :

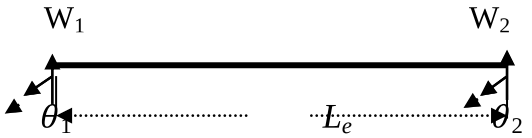

Fig. 2 Finite element degrees of freedom: equivalent beam model.

$$N_1 = 1 - \frac{3y_e^2}{L_e^2} + 2\,\frac{y_e^3}{L_e^3}\,, N_2 = y_e - \frac{2y_e^2}{L_e} + \frac{y_e^3}{L_e^2}\,, N_3 = \frac{3y_e^2}{L_e^2} - 2\,\frac{y_e^3}{L_e^3}\,, N_4 = \frac{y_e^3}{L_e^2} - \frac{y_e^2}{L_e}$$

Applying the principle of stationary potential energy, the following matrix equation is obtained [5].

$$\left([K^e] - \hat{P}_{cr}\,[K_G^e]\right)\{U^e\} = \{0\} \tag{7}$$

where, $\{U^e\}$ is the nodal displacement vector, $[K^e]$ and $[K_G^e]$ are the stiffness and geometric matrices, respectively normalized w.r.t the bending stiffness EI_0 of the baseline design. The stiffness matrix is defined as:

$$[\mathrm{K^e}] = \begin{bmatrix} 12\mathrm{K_b}/\hat{\mathrm{L}}_\mathrm{e}^2 & 6\mathrm{K_b}/\hat{\mathrm{L}}_\mathrm{e} & -12\mathrm{K_b}/\hat{\mathrm{L}}_\mathrm{e}^2 & 6\mathrm{K_b}/\hat{\mathrm{L}}_\mathrm{e} \\ 6\mathrm{K_b}/\hat{\mathrm{L}}_\mathrm{e} & 4\mathrm{K_b} & -6\mathrm{K_b}/\hat{\mathrm{L}}_\mathrm{e} & 2\mathrm{K_b} \\ -12\mathrm{K_b}/\hat{\mathrm{L}}_\mathrm{e}^2 & -6\mathrm{K_b}/\hat{\mathrm{L}}_\mathrm{e} & 12\mathrm{K_b}/\hat{\mathrm{L}}_\mathrm{e}^2 & -6\mathrm{K_b}/\hat{\mathrm{L}}_\mathrm{e} \\ 6\mathrm{K_b}/\hat{\mathrm{L}}_\mathrm{e} & 2\mathrm{K_b} & -6\mathrm{K_b}/\hat{\mathrm{L}}_\mathrm{e} & 4\mathrm{K_b} \end{bmatrix} \tag{8}$$

where K_b is the normalized stiffnesses defined as follows :

$$\mathrm{K_b} = \frac{\widehat{\mathrm{EI}}}{\hat{\mathrm{L}}_\mathrm{e}} \quad , \quad \widehat{\mathrm{EI}} = \frac{\mathrm{EI}}{\mathrm{EI_0}} \quad , \quad \hat{\mathrm{L}}_\mathrm{e} = \frac{\mathrm{L_e}}{\mathrm{L_0}} \tag{9}$$

EI_0 is the bending stiffness of the baseline design, and L_0 is the total length of the baseline design. The geometric matrix is defined as [5]:

$$[\mathrm{K_G^e}] = \begin{bmatrix} 1.2/\hat{\mathrm{L}}_\mathrm{e} & 0.1 & -1.2/\hat{\mathrm{L}}_\mathrm{e} & 0.1 \\ 0.1 & 0.4\hat{\mathrm{L}}_\mathrm{e}/3 & -0.1 & -0.1\hat{\mathrm{L}}_\mathrm{e}/3 \\ -1.2/\hat{\mathrm{L}}_\mathrm{e} & -0.1 & 1.2/\hat{\mathrm{L}}_\mathrm{e} & -0.1 \\ 0.1 & -0.1\hat{\mathrm{L}}_\mathrm{e}/3 & -0.1 & 0.4\hat{\mathrm{L}}_\mathrm{e}/3 \end{bmatrix} \tag{10}$$

The entire or global stiffness matrices of the assembled column can be obtained by summing up the individual matrices of each element. Applying the boundary conditions, the associated eignvalue problem of column buckling is described by the following matrix equation:

$$\frac{1}{\hat{P}_{cr}}\{U\} = [K]^{-1}[K_G]\{U\} \tag{11}$$

where:

$$\hat{P}_{cr} = \frac{PL_0^2}{EI_0} \tag{12}$$

$[K]$ and $[K_G]$ are the global stiffness and geometric matrices respectively. The critical buckling load corresponds to the largest eignvalue $\frac{1}{\hat{P}_{cr}}$.

3. Formulation of the Buckling Optimization Problem

The present optimization problem seeks column designs having the largest possible critical (lowest) buckling load, while preserving the total structural mass at a specified value. In the present formulation, the preassigned parameters, which do not change through the design process, are chosen to be the type and location of supports, shape of the cross section, type of materials of construction and the total column's length. All these parameters define a baseline design having uniform properties lengthwise. To accurately define the true design variables that have a direct bearing on buckling optimization, let us first examine, as a fundamental case study, a uniform cantilevered column composed of one segment (N_s = 1). Lowest (critical) buckling load given by:

$$\hat{P}_{cr} = \frac{\pi^2 \hat{E}_1 \hat{I}_1}{4\hat{L}_1^2} \;,\; (\hat{I}_1 = \hat{h}_1), \text{ for thin-walled columns.} \tag{13}$$

On the other hand, the nondimensional structural mass is expressed as (see Table 1).

$$\widehat{M}_s = \hat{\rho}_1 \hat{h}_1 \hat{L}_1 \tag{14}$$

It is obvious that the main design variables affecting buckling optimization are the mass density, modulus of elasticity, wall thickness and the segment length. In composite structures, however, the density and modulus of elasticity depend on the volume fraction (V), of the constituent materials.

This problem can be easily solved by the well-established unconstrained mathematical programming techniques [18], with the elimination of one of the design variables using the explicit expression of the mass equality constraint. Equation. (13) may be thought of as an explicit function describing the critical buckling load in terms of the design variables. It is noticed that $\hat{P}_{cr}$ increases monotonically with $\hat{E}_1$ and $\hat{h}_1$ and decreases with $\hat{L}_1$, which is a natural expected behavior. Therefore, instead of treating $\hat{P}_{cr}$ as an implicit function, it is possible to choose prescribed values for it and either $\hat{E}_1$ or $\hat{L}_1$, and solve Eq. (12) numerically for the remaining unknown variable, which must also satisfy Eqs. (13) and (14).

The above mathematical fundamentals will be confirmed and applied to columns composed of more than one segment. The finite element analysis outlined in section 2 can be coupled with a standard nonlinear mathematical programming algorithm for the search of column designs with the largest possible resistance against buckling. Therefore, the strongest column design problem may be cast in the following optimization problem:

$$
\begin{aligned}
&\text{Minimize} \quad -\hat{P}_{cr}\left(\{\mathrm{V}, \hat{h}, \hat{L}\}_{k=1,2,\ldots,N_s}\right)\\
&\text{Subject to} \quad \hat{M}_s = 1\\
&\qquad\qquad\quad \textstyle\sum_{k=1}^{N_s} \hat{L}_k = 1
\end{aligned}
\qquad (15)
$$

Such an optimization problem can be further simplified by eliminating any two of the design variables using the mass and length equality constraints. However, side constraints are always needed to impose lower and upper limits on the design variables to avoid having odd-shaped unrealistic column designs. Iterative techniques, such as the sequential quadratic programming [18], are usually used to obtain the needed optimal solutions, in which a series of directed design changes (moves) are made between successive points in the design space.

Three problems having different concepts of axial grading will be treated next. The first one considers axial material grading of composite column structures made of two different materials, making the physical and mechanical properties change in the axial direction. The second problem considers piecewise axial grading of the cross-section wall thickness of unidirectional composite columns with the associated design variables selected to be the wall thickness and length of each segment composing the column. The third problem combines material and thickness grading together. In all cases, the optimization problem is formulated in a dimensionless form, making the model independent of any specific cross-sectional shape or dimensions.
Baseline design parameters: L = total column's length, h = wall thickness, I = second moment of area, E = modulus of elasticity, ρ = mass density.

4. Columns with Axial Material Grading

Composite columns made of two different materials denoted by (A) and (B) will be considered herein. The physical and mechanical properties are allowed to vary lengthwise, yielding a grading of the material in the direction of the column's axis. The distributions of the mass density ρ and modulus of elasticity E are determined by utilizing Halpin and Tsai semi-empirical formulas (see Ref 19). Assuming no voids are present, we have:

Volume fractions: $V_A(x) + V_B(x) = 1$ $\qquad (16a)$
Mass density: $\rho(x) = V_A(x)\rho_A + V_B(x)\rho_B$ $\qquad (16b)$
Modulus of elasticity: $E(x) = V_A(x)E_A + V_B(x)E_B$ $\qquad (16c)$

Table 1 Definition of dimensionless quantities.

Quantity	Nondimensionalization
Axial coordinate	$\hat{y} = y/L$
Length of K th segment	$\hat{L}_k = L_k/L$
Transverse deflection	$\hat{w} = w/L$
Wall thickness	$\hat{h}_k = h_k/h$
Second moment of area	$\hat{I}_k = I_k/I(= \hat{h}_k)$
Modulus of elasticity	$\hat{E}_k = E_k/E$
Bending moment	$\hat{M} = M * (\frac{L}{EI})$
Shearing force	$\hat{F} = F * (\frac{L^2}{EI})$
Axial force	$\hat{P} = P * (\frac{L^2}{EI})$
Mass density	$\hat{\rho}_k = \rho_k/\rho$
Total structural mass	$\hat{M}_s = \sum_{k=1}^{N_s} \hat{\rho}_k \hat{h}_k \hat{L}_k$

The baseline design having uniform mass and stiffness distributions is chosen to be constructed from the same type of composite material with equal volume fractions of its constituents, i.e. $V_{A0} = V_{B0} = 50\%$. It has also the same type, peripheral dimensions and wall thickness of the cross section. i.e. $\hat{I} = \hat{h} = 1$. Therefore, the mass density and modulus of elasticity of the baseline design are given by:

$$\rho_0 = \frac{\rho_A + \rho_B}{2} \qquad (17a)$$

$$E_0 = \frac{E_A + E_B}{2} \qquad (17b)$$

Therefore, the corresponding dimensionless quantities for the Kth segment can be determined from the relations (refer to Table 1):

$$\hat{\rho}_K = \frac{2(\rho_A V_{A,K} + \rho_B V_{B,K})}{\rho_A + \rho_B}, \quad k = 1,2, \ldots\ldots, N_s \qquad (18a)$$

$$\hat{E}_K = \frac{2(E_A V_{A,K} + E_B V_{B,K})}{E_A + E_B}, \quad k = 1,2, \ldots\ldots, N_s \qquad (18b)$$

Since the total structural mass M_s is kept equal to that of the baseline design, then a feasible design should satisfy the following dimensionless mass equality constraint:

$$\sum_{k=1}^{N_s} V_{A,K}\hat{L}_K = 0.5 \tag{19}$$

The above mathematical model will now be applied to the case of a cantilevered column constructed from unidirectional fibrous composites with the properties given in Table 2.

Table 2 Material properties of selected fiber-reinforced composites [19].

Composite material	Material (A)= fibers		Material (B)= matrix	
	$\rho_A(\frac{g}{cm^3})$	E_A (Gpa)	$\rho_B(\frac{g}{cm^3})$	E_B(Gpa)
E-glass/epoxy	2.54	73	1.27	4.3
S-glass/epoxy	2.49	86	1.27	4.3
Carbon/epoxy	1.81	235	1.27	4.3
E-glass/Vinylester	2.54	73	1.15	3.5
S-glass/Vinylester	2.49	86	1.15	3.5

For a cantilevered column, the boundary conditions are:

At $y = 0$ $\quad \hat{w} = \theta = 0 \quad (20a)$
At $y = 1$ $\quad \hat{M} = \hat{F} = 0 \quad (20b)$

Optimum cantilevered columns made of unidirectional E-glass/epoxy composites having the largest possible resistance against buckling are given in Table 3, which compares the present result with those given in Ref. [20]:

Table 3 Axial material grading

No. of segments	Present	Ref [20]
One segment	$(\hat{P}_{cr})_{max}$ =2.4674 $(V_A,\hat{L})_{k=1}$ (0.5,1)	$(\hat{P}_{cr})_{max}$ =2.4674 $(V_A,\hat{L})_{k=1}$ (0.5,1)
Two segments	$(\hat{P}_{cr})_{max}$ =2.8215 $(V_A,\hat{L})_{k=1,2}$ (0.6395,0.6826),(0.20,0.3174)	$(\hat{P}_{cr})_{max}$ =2.8227 $(V_A,\hat{L})_{k=1,2}$ (0.6375,0.6875),(0.1975,0.3125)
Three segments	$(\hat{P}_{cr})_{max}$ =2.8847 $(V_A,\hat{L})_{k=1,2,3}$ (0.6935,0.4653),(0.4417,0.2911),(0.20,0.2436)	$(\hat{P}_{cr})_{max}$ =2.911 $(V_A,\hat{L})_{k=1,2,3}$ (0.70,0.514),(0.4125,0.2785),(0.122,0.2075)

The developed contours of the dimensionless critical buckling load $\hat{P}_{cr}$ augmented with the equality mass constraint ($\hat{M}_s = 1$) are shown in Fig. 3. The global optimal solution can be found to be $(\hat{P}_{cr})_{max}$ =2.8215, occurring at the design point $(V_A,\hat{L})_{k=1,2}$ =(0.6395,0.6826),(0.20,0.3174). This means that the strongest column made of only two segments can withstand a buckling load 14.35% higher than that with uniform mass and stiffness distributions.

To verify the results, the standard computer package ANSYS was applied to a cantilevered column divided into 15 element with V_f = 50%. The obtain result $\widehat{P}_{cr}$ = 2.467 is the same as that obtained using MATLAB code. Fig. 4 shows the first buckling mode of the optimum cantilevered design.

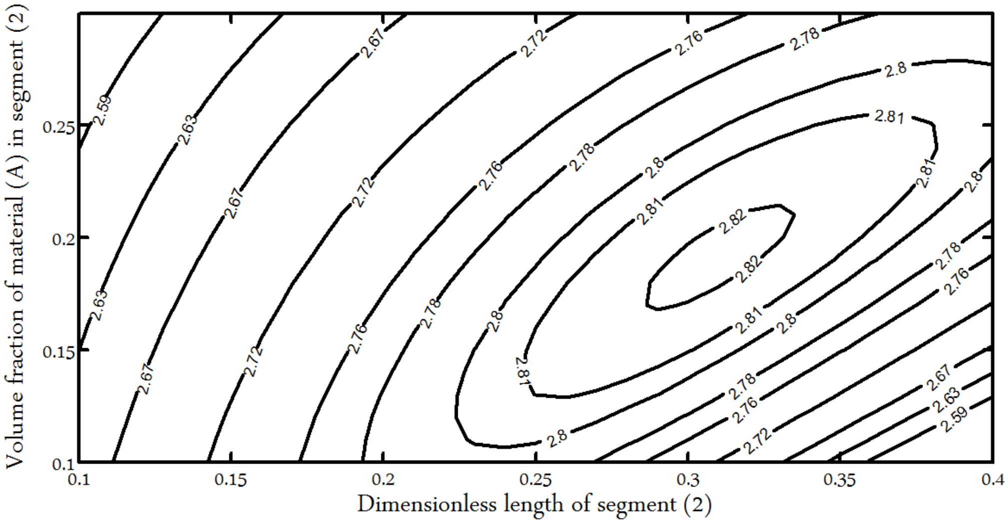

Fig. 3 $\widehat{P}_{cr}$-contours for a 2-segment cantilevered column made of E-glass/epoxy. Design space with total structural mass preserved, $\widehat{M}_s = 1$.

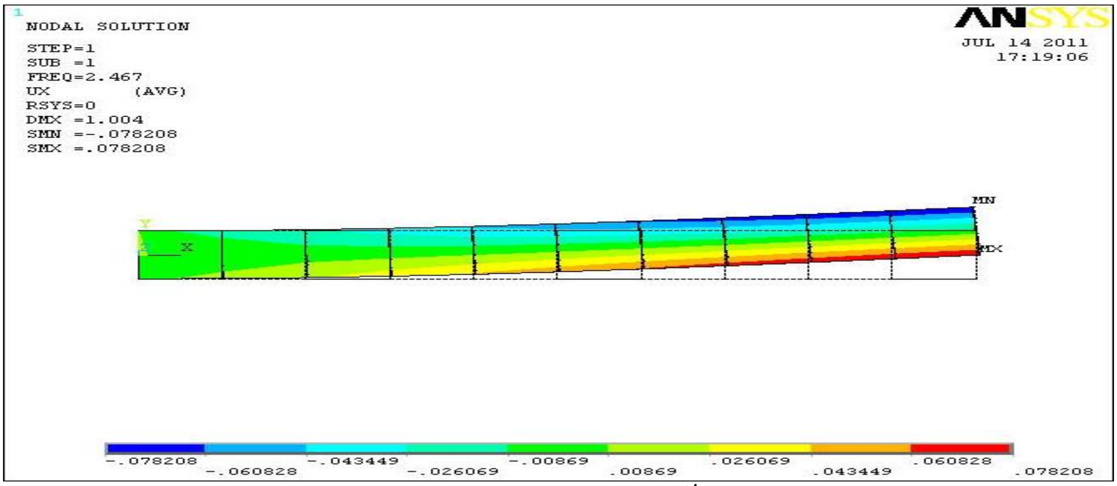

Fig. 4 ANSYS salutation showing 1st mode buckling .

5. Thin-Walled Columns with Thickness Grading

Thin-walled tubular sections are more economical than solid sections for compression members. By optimizing the wall thickness, the overall stability level can be substantially improved without the penalty of increasing structural weight. There is a lower limit for the wall thickness, below which the wall itself becomes unstable, and instead of buckling of the column as a whole, there occurs a type of buckling which brings about corrugation of the wall. This condition requires the analysis of cylindrical shell buckling, which is beyond the scope of the present study.

Librescu and Maalawi [21] considered aeroelastic design optimization of subsonic wings by grading the wall thickness of the cross section. The optimization problem was solved by the interior penalty function technique coupled with the associated eigenvalue routines. It was shown that good patterns with decreasing wall thickness towards the wing tip produce significant improvement in the overall torsional stability level of the wing.

The same mathematical consequences developed for axial material grading can also be applied herein. The resulting optimum patterns for the strongest cantilevered columns are shown in Table. 4. The attained optimization gain has reached a value of 14.35% for a 2-segment column. The optimal segment lengths were found to be similar to those obtained for the case of material grading. Table 4 compares the present result with those given in Ref. [20]

Table 4 Wall thickness grading

Number of segments	Present	Ref [20]
One segment	$(\hat{P}_{cr})_{\max}$ =2.4674 $(\hat{h}_k, \hat{L})_{k=1}$ (1,1)	$(\hat{P}_{cr})_{\max}$ =2.4674 $(\hat{h}_k, \hat{L})_{k=1}$ (1,1)
Two segments	$(\hat{P}_{cr})_{\max}$ =2.8216 $(\hat{h}_k, \hat{L})_{k=1,2}$ (1.2436,0.6949),(0.445,0.3051)	$(\hat{P}_{cr})_{\max}$ =2.8227 $(\hat{h}_k, \hat{L})_{k=1,2}$ (1.2436,0.6875),(0.45,0.3125)
Three segments	$(\hat{P}_{cr})_{\max}$ =2.9109 $(\hat{h}_k, \hat{L})_{k=1,2,3}$ (1.3154,0.5926),(0.7258,0.244), (0.2655,0.1634)	$(\hat{P}_{cr})_{\max}$ =2.911 $(\hat{h}_k, \hat{L})_{k=1,2,3}$ (1.3547,0.514),(0.846,0.2785), (0.3325,0.2075)

It is to be noticed here that the material of construction has been chosen to be isotropic, making the dimensionless modulus of elasticity and mass density equal to unity, i.e. $\hat{E}_k = \hat{\rho}_k = 1$, k=1,2,000,$N_s$. Therefore:

$$p_k = \sqrt{\hat{p}/\hat{h}_k} \qquad k = 1,2, \ldots\ldots, N_s \tag{21a}$$

Where the mass equality constraint becomes .

$$\hat{M}_s = \sum_{k=1}^{N_s} \hat{h}_k \hat{L}_k = 1 \tag{21b}$$

As in the case of material grading, the total number of design variables can also be reduced to 2(N_s – 1). One of the segment lengths can be eliminated because of the equality constraint $\sum \hat{L}_k = 1$. Another variable can also be discarded by applying the mass equality constraint of Eq. (21b). This reduces the dimensions of the associated optimization problem by two, which can yield a significant saving in the computational time.

The developed contours of the dimensionless critical buckling load $\hat{P}_{cr}$ augmented with the equality mass constraint ($\hat{M}_s = 1$) are shown in Fig. 5. The global optimal solution was found to be $(\hat{P}_{cr})_{\max}$ =2.8216, occurring at the design point $(\hat{h}_k, \hat{L})_{k=1,2}$ =(1.2436,0.6949),(0.445,0.3051). This means that the strongest column made of only two segments can withstand a buckling load 14.35% higher than that with uniform mass and stiffness distributions.

6. Combined Material and Thickness Grading

The combined grading consider volume fraction, length and wall thickness variation in the axial direction. By optimizing the volume fraction, length and wall thickness the overall stability level can be substantially improved without the penalty of increasing structural weight. Table 5 presents the results of this case. The global optimal solution was found to be $(\hat{P}_{cr})_{max}$ =3.6017, occurring at the design point

$(V_A, \hat{h}_k, \hat{L})_{k=1,2}$ = (0.8,1.0199,0.7198),(0.8,0.354,0.2802).

This means that the strongest column made of only two segments can withstand a buckling load 45.97% higher than that with uniform mass and stiffness distributions.

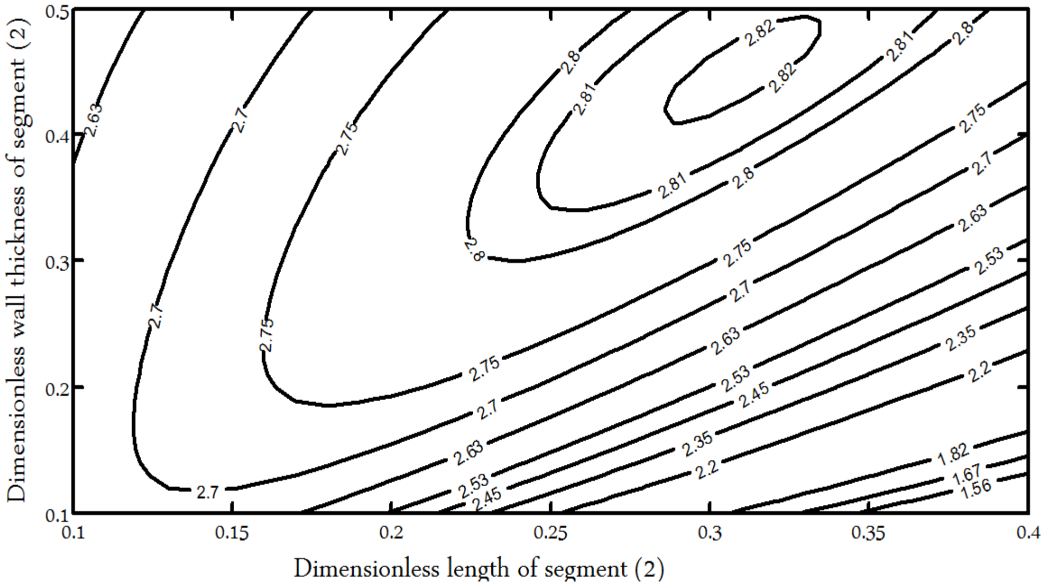

Fig. 5 $\hat{\mathbf{P}}_{cr}$-contours for a 2-segment cantilevered column made of E-glass/epoxy. Design space with total structural mass preserved, $\widehat{\boldsymbol{M}}_s = \mathbf{1}$.

Table 5 Combined material and thickness grading

Number of segments	Combination grading
One segment	$(\hat{P}_{cr})_{max}$ =3.1526 $(V_A, \hat{h}_k, \hat{L})_{k=1}$, (0.8,0.8333,1)
Two segments	$(\hat{P}_{cr})_{max}$ =3.6017 $(V_A, \hat{h}_k, \hat{L})_{k=1,2}$ (0.8,1.0199,0.7198),(0.8,0.354,0.2805)
Three segments	$(\hat{P}_{cr})_{max}$ =3.7186 $(V_A, \hat{h}_k, \hat{L})_{k=1,2,3}$ (0.8,1.1093,0.5857),(0.8,0.5939,0.2616),(0.8,0.1851,0.1527)

7. Dual Problem of Minimizing the Total Structural Mass

Table 6 presents the dual problem of minimizing the total structural mass:

Minimize $\hat{M}_s$ ($\{V, \hat{L}\}_{k=1,2,\ldots,N_k}$)
Subject to $\hat{P}_{cr} \geq \hat{P}_{cr0}$

$$\sum_{k=1}^{N_s} \hat{L}_k = 1 \qquad (22)$$

It is seen that the optimal column design with two segments and having the same critical buckling load of the baseline design is 11% lighter weight.

Table 6 Dual problem of minimizing the total structural mass

Number of segments	Axial material grading
Two segments	$(\hat{M}_s)$ =0.89 $(V_A, \hat{L})_{k=1,2}$ (0.633,0.6829),(0.18,0.3171)

8. Conclusions

Three different approaches of using the concept of axial grading for the enhancement of buckling stability of slender, elastic columns have been presented. The first one considers axial material grading in which the volume fraction of two material constituents are chosen to vary piece wisely in the direction of the column axis. This allows the physical and mechanical properties of the material to be tailored in order to maximize the critical buckling load while maintaining the total structural mass constant. The second problem consider piecewise axial grading of the cross section wall thickness of unidirectional composite columns constructed from uniform segments. The associated design variables have been selected to be the wall thickness and length of each segment composing the column. The design variables of the third problem were chosen to be the volume fraction, wall thickness and length of each segment.
Numerical results of the critical buckling load for one, two and three segments in the different cases are presented and discussed.

The following conclusions are obtained:

- The combined grading gives the maximum value of the critical buckling load.
- The exact buckling load is obtained for any number of segments, type of cross section and type of boundary conditions.
- The present multi-segment model has the advantage of achieving global optimality for the strongest column shapes that can be fabricated economically from any arbitrary number of uniform segments.
- The increase in the number of segment would naturally result in an increase in the maximum buckling load, however, this would certainly increase the production and manufacturing costs.

Caution regarding the design of beam-columns having discontinuous section dimensions or material properties must be considered, since stress concentrations may arise at such discontinuities. Future studies may apply the same approach outlined herein to problems of structural dynamic optimization of functionally graded material beams. Sensitivity of the design variables to the buckling load should be included in a more general formulation. The method can be also extended to cover stability and dynamic optimization of several types of thin-walled structures with either material or thickness grading along predetermined directions.

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